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SUB: Engineering mathematics - I

unit - I

Algebra - Determinants

Defn: Determinant is a square arrangement of numbers Real (or) complex with in two vertical lines.

Order of the determinants It is defined as the number of rows in the determinants

Example $\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$ are determinants of order 2.

Example Find the value of $\begin{vmatrix} 1 & -5 & 6 \\ 7 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix}$

Solution:-

Let $\Delta = \begin{vmatrix} 1 & -5 & 6 \\ 7 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix}$

$= 1\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 7 & 4 \\ 3 & 1 \end{vmatrix} + 6\begin{vmatrix} 7 & 3 \\ 3 & 2 \end{vmatrix}$

$\boxed{\Delta = 0}$

Properties of determinants

If any two rows and colums of a determinants are Interchanged, then the value of the determinants is changed by its sign

then $\boxed{\Delta = -\Delta'}$

ii) minor of an element The minor of an element is a determinant obtained by deleting the row and column in which the element occurs

Example:- Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

i) Minor of $a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2 c_3 - b_3 c_2$

Solution of system of linear equations by determinant method

Crammer's rule

Solution is $\boxed{x = \dfrac{\Delta x}{\Delta}, \quad y = \dfrac{\Delta y}{\Delta}, \quad z = \dfrac{\Delta z}{\Delta}}$

Examples:-

$x - y = 1, \quad 2x - 3y + 1 = 0$

Given:-

$x - y = 1$

$2x - 3y = 1$

$$\Delta = \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} = -3 + 2 = -1$$

$$\Delta x = \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} = -3 - 1 = -4$$

$$\Delta y = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -2$$

$$x = \frac{\Delta x}{\Delta} = \frac{-4}{-1} = 4, \qquad y = \frac{\Delta y}{\Delta} = \frac{-3}{-1} = 3$$

Examples:- solve by using cramer's rule

$x + y + z = 3, \quad 2x - y + z = 2, \quad 3x + 2y - 2z = 3$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= 7 + 7 = 14$$

$$\Delta x = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{vmatrix} = 3\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= 7 + 7 = 14$$

III^ly

$$\Delta y = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & -2 \end{vmatrix} = -14$$

$$x = \frac{\Delta x}{\Delta}, \quad y = \frac{\Delta y}{\Delta}, \quad z = \frac{\Delta z}{\Delta}$$

$$\Delta z = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 14$$

$$\boxed{x = \frac{14}{14} = 1, \quad y = \frac{14}{14} = 1, \quad z = \frac{14}{14} = 1}$$

Matrices

Defn The rectangular array of numbers arranged in rows and columns enclosed within the bracket is called a matrix.

Example

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & -1 & 2 \\ 5 & 6 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 7 & 5 \\ -1 & 0 \\ 2 & 5 \end{pmatrix}$$

Order of Matrix

If a Matrix "m" rows "n" columns

Example

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 5 \\ 3 & 7 \end{pmatrix} \quad \text{order} = 3 \times 2$$

Types of Matrix

1. **Row Matrix** Example :- $(3 \ -2 \ 7)$ order 1×3

2. **column, matrix** a Matrix having oney one column

$$\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} \quad \text{order} = 3 \times 1$$

3. **Square Matrix** Example: $\begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix}$

4. **Zero Matrix** in the Matrix if all elements are zero

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. **Transpose of a Matrix**

$$\text{Let } A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\text{Transpose of } A^T = \begin{pmatrix} -1 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{pmatrix}$$

6. **symmetric Matrix** :- Let $A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

Here $\boxed{A^T = A}$

7. **Unit Matrix**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (or) } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Definition</u> <u>singular matrix</u> A square matrix is said to be singular matrix if its determinant value is zero

$$\boxed{|A| = 0}$$

<u>Example</u> let $A = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{vmatrix}$

$$\boxed{|A| = 0}$$

<u>Inverse of a Matrix</u>:- let A be a square matrix, If there exists a square matrix B such that $AB = BA = I$, then B is called the inverse of "A" and it is denoted by A^{-1}

$$\boxed{A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)}$$

Find the Inverse of Matrix $\begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{pmatrix}$

<u>Step: 1</u>

$$|A| = \begin{vmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{vmatrix} = 3(-6+4) - 4(0-10) + 1(0+5)$$

$$= -6 + 40 + 5 = 39$$

<u>Step: 2</u> cofactor of $3 = -2 = \begin{vmatrix} -1 & 2 \\ -2 & 6 \end{vmatrix} = (-2)$

<u>III^ly</u> cofactor of $4 = 10$, cofactor of $0 = -26$

cofactor of $-1 = 13$ cofactor $2 = 26$

cofactor of $5 = 9$ cofactor of $-2 = -6$

cofactor of $6 = -3$

$$\text{cofactor of Matrix} = \begin{pmatrix} -2 & 10 & 5 \\ -26 & 13 & 26 \\ 9 & -6 & -3 \end{pmatrix}$$

$$\operatorname{Adj}(A) = \begin{pmatrix} -2 & -26 & 9 \\ 10 & 13 & -6 \\ 5 & 26 & -3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{39} \begin{bmatrix} -2 & -26 & 9 \\ 10 & 13 & -6 \\ 5 & 26 & -3 \end{bmatrix}$$

Rank of Matrix

Defn i) A has atleast one minor of order r which not vanish.

ii) Every minor of A of order (r+1) and higher order Vanishes

(ie) The rank of Matrix is the the order of any highest order non-vanishing minor of the Matrix.

The rank of "A" is denoted by $\rho(A)$

Find the rank of Matrix $\begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$

The minor is given by $\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix} = 7+2 = 9 \neq 0$

The highest order of non-vanishing minor of A is 2//

Binomial theorem

Some formulas

i) Factorial $n!$ $0! = 1$ $1! = 1$

ii) permutation $nP_r = \dfrac{n!}{(n-r)!} = 7P_3 = 5\times6\times7 = 210$ //

iii) compination. $nC_r = \dfrac{n!}{(n-r)!\ r!} = 5C_3 = 10$ //

IV) Binomial theorem (positive Integral Index)

$(x+a)^n = x^n + nC_1 x^{n-1} a^2 + nC_2 x^{n-2} a^2 + \ldots$

v) write the General term $T_{r+1} = nC_r x^{n-r} a^r$

vi) Middle term $\dfrac{n+1}{2}, \dfrac{n+3}{2}$.

i) **How many Middle term** in the expansion $\left(x+\dfrac{3}{x^2}\right)^{13}$

Solu $n = 13$ odd number

There are two middle terms.

Some problems

① Find the general term in the expansion of $\left(2x - \frac{1}{x}\right)^7$.

Formulas $\boxed{T_{r+1} = nC_r \, x^{n-r} \, a^r}$ $x = 2x$ $a = \frac{-1}{x}$ $n = 7$ $r = r$

$$T_{r+1} = 7C_r \, (2x)^{7-r} \left(\frac{-1}{x}\right)^r$$

$$= 7C_r \, 2^{7-r} \, x^{7-r} \frac{(-1)^r}{x^r}$$

$$\boxed{T_{r+1} = 7C_r \, 2^{7-r} \, (-1)^r \, x^{7-2r}}$$

② Find the 7^{th} term in the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{10}$

Solu $\boxed{T_{r+1} = nC_r \, x^{n-r} \, a^r}$ $x = x^2$ $a = \frac{-1}{x^2}$ $n = 10$ $r = 6$

$$T_{6+1} = 10C_6 \, (x^2)^{10-6} \left(-\frac{1}{x^2}\right)^6$$

$$= 10C_6 \, (x^2)^4 \frac{(-1)^6}{(x^2)^6}$$

$$= 10C_6 \, x^8 \frac{(-1)^6}{x^{12}}$$

$$= 10C_6 \, (-1)^6 \, x^{-4}$$

$$\boxed{T_7 = 10C_6 \, x^{-4}}$$

③ Find the middle term in the Expansion $\left(x^4 - \frac{1}{x^3}\right)^{16}$

Formula $\boxed{T_{r+1} = nC_r \, x^{n-r} \, a^r}$ $x = x^4$ $a = \frac{-1}{x^3}$ $n = 16$

even number

$\boxed{r = \frac{n}{2} + 1} = \frac{16}{2} + 1 = 9^{th}$

$$T_{8+1} = 16C_8 \, (x^4)^{16-8} \left(\frac{-1}{x^3}\right)^8$$

$$= 16C_8 \, (x^4)^8 \frac{(-1)^8}{(x^3)^8}$$

$$= 16C_8 \, x^{32} \, (-1)^8 / x^{24}$$

$$= 16C_8 \, (-1)^8 \, x^{32-24}$$

$$\boxed{T_9 = 16C_8 \, (-1)^8 \, x^8}$$

(4) Find the middle term in the expansion $\left(2x+\frac{1}{x}\right)^{13}$

Formula $\boxed{T_{r+1} = nC_r \, x^{n-r} \, a^r}$

$x = 2x \quad a = \frac{1}{x} \quad n = 13$

$r =$ odd number

$\frac{n+1}{2} = \frac{13+1}{2}, \quad \frac{n+3}{2} = \frac{13+3}{2}$

$= \frac{14}{2} = 7, \quad \frac{16}{2} = 8$

term term

$T_{6+1} = 13C_6 (2x)^{13-6} \left(\frac{1}{x}\right)^6$

$= 13C_6 (2x)^7 \left(\frac{1^6}{x^6}\right)$

$= 13C_6 \, 2^7 \, x^7 \, 1^6 \, x^{-6}$

$\boxed{T_7 = 13C_6 \, 2^7 \, x^1}$

Middle term 8^{th} term

$T_{7+1} = 13C_7 (2x)^{13-7} \left(\frac{1}{x}\right)^7$

$= 13C_7 (2x)^6 \, \frac{1^7}{x^7}$

$= 13C_7 \, 2^6 \, x^6 \, 1^7 \, x^{-7}$

$\boxed{T_8 = 13C_7 \, 2^6 \, x^{-1}}$

(5) Find the co-efficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

$\boxed{T_{r+1} = nC_r \, x^{n-r} \, a^r}$

$x = x^4 \quad a = \frac{-1}{x^3} \quad n = 15$

$= 15C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r$ $\quad r = r$

$= 15C_r \, x^{60-4r} \frac{(-1)^r}{x^{3r}}$

$= 15C_r \, x^{60-4r} (-1)^r x^{-3r}$

$= 15C_r (-1)^r x^{60-4r-3r}$

$\boxed{T_{r+1} = 15C_r (-1)^r x^{60-7r}}$

Find the co-efficient of x^{32}

$x^{60-7r} = x^{32} \Rightarrow 60-7r = 32$

$60-32 = 7r$

$28 = 7r$

$\boxed{r = 4}$

Put $r = 4$

$\therefore T_{4+1} = 15C_4 (-1)^4 x^{60-7(4)}$

$= 15C_4 (-1)^4 x^{60-28}$

$\boxed{T_5 = 15C_4 (-1)^4 x^{32}}$

⑥ Find the Independent of "x" in the expansion $\left(2x^2 - \frac{1}{x}\right)^{12}$.

<u>Solution</u>

$x = 2x^2 \quad a = \frac{-1}{x} \quad n = 12 \quad r = r$

$$\boxed{T_{r+1} = nC_r \, x^{n-r} \, a^r}$$

$$= 12C_r \, (2x^2)^{12-r} \left(\frac{-1}{x}\right)^r$$

$$T_{r+1} = 12C_r \, 2^{12-r} \, (x^2)^{12-r} \, \frac{(-1)^r}{x^r}$$

$$= 12C_r \, 2^{12-r} \, (-1)^r \, x^{24-2r} \, x^{-r}$$

$$= 12C_r \, 2^{12-r} \, (-1)^r \, x^{24-2r-r}$$

$$\boxed{T_{r+1} = 12C_r \, 2^{12-r} \, (-1)^r \, x^{24-3r}}$$

To find term of independent of x

$$x^0 = x^{24-3r}$$

$$24 - 3r = 0$$

$$24 = 3r$$

$$\boxed{r = \frac{24}{3} = 8}$$

put $\boxed{r = 8}$

$$T_{8+1} = 12C_8 \, 2^{12-8} \, (-1)^8 \, x^{24-3(8)}$$

$$= 12C_8 \, 2^4 \, x^0 \,//$$

The term of independent of $x = 12C_8 \, 2^4 \,//$

Unit - II

Complex Numbers

Defn.

A number of the form $a+ib$ where a, b are real numbers and $i^2=-1$ is called a complex Number. Then first term "a" is called the real part and "b" is called Imaginary part of complex numbers.

It is denoted by $z = a+ib$

complex conjugate:- $a-ib$ is said to be conjugate to each other.

Algebra of complex Numbers

1. Equal complex Numbers Ex: $z_1 = x_1+iy_1$

$z_2 = x_2+iy_2$

2. Negative complex Numbers $-z = -x+i(-y)$
3. Sum (or) difference of two complex numbers

$$z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$$

4. Product of complex numbers

$$z_1z_2 = (x_1+iy_1)(x_2+iy_2)$$

5. quotient of two complex numbers

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2}$$

Worked Examples:-

1. Find the value of $i^2+i^3+i^4$

$$I = -1+(-1)i+(i^2)^2$$

$$= -1-i+1$$

$$= -i$$

2. Find the real and imaginary part of $\frac{1+2i}{3}$

Real part : $\frac{1}{3}$

Imaginary part: $2/3$

2. $z_1 = 1+i$, $z_2 = 3+2i$ find $3z_1 + 4z_2$

Given $z_1 = 1+i$

$z_2 = 3+2i$

$3z_1 + 4z_2 = 3+3i+12+8i$

$= 15+11i$

3. Express in $1+i$ in modulus and amplitude form

Modulus $r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$

amplitude $\theta = \tan^{-1}(b/a) = \tan^{-1}(1/1)$

$= \pi/4$

4. Find the real and imaginary part $\frac{(1+i)(2-i)}{(1-i)^2}$

$$z = \frac{(1+i)(2-i)}{(1-i)^2}$$

$$= \frac{(1+i)(2-i)}{(1-i)(1-i)}$$

$$= \frac{2-i+2i-i^2}{1-i-i+i^2}$$

$$= \frac{3+i}{-2i}$$

$$\frac{3+i}{-2i} \times \frac{2i}{2i} = \frac{3+i}{-2i} \times \frac{i}{i}$$

$$= \frac{3i+i^2}{-2i^2}$$

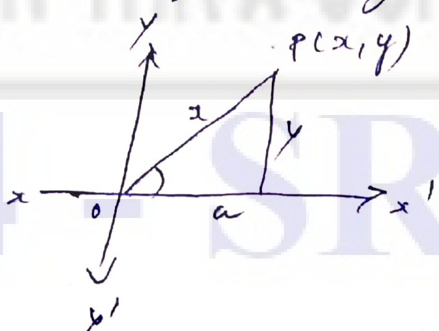
$$= \frac{3i+(-1)}{-2(-1)}$$

$$= \frac{-1}{2} - \frac{3}{2}i$$

Real part : $-1/2$

Imaginary part : $-3/2$

5. Argand plane (Argand diagram)



x - axis considered real axis

Y - axis consided as imaginary axis

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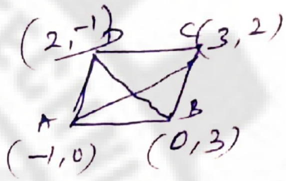
distance between two complex Numbers

$z_1 = x_1 + iy_1 \quad : \quad z_2 = x_2 + iy_2$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. Prove that the points representing the complex numbers $-1, 3i, 3+2i, 2-i$ form a square.



$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(0+1)^2 + (3-0)^2}$$

$$= \sqrt{1+9}$$

$$\boxed{AB = \sqrt{10}}$$

$$BC = \sqrt{(3-0)^2 + (2-3)^2}$$

$$= \sqrt{3^2 + (1)^2} = \sqrt{9+1}$$

$$\boxed{BC = \sqrt{10}}$$

IIIly

$CD = \sqrt{10}$

$DA = \sqrt{10}$

Distance: $AC = \sqrt{20}$

$BD = \sqrt{20}$

$\therefore AB = BC = CD = DA = \sqrt{10}$

$AC = BD = \sqrt{20}$

2. Prove that $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right)^4 = 1$

Solution:-

$$z = \frac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}$$

$$= \frac{i(\cos\theta + i\sin\theta)}{i(\sin\theta - i\cos\theta)}$$

$$= \frac{i(\cos\theta + i\sin\theta)}{\cos\theta + i\sin\theta}$$

$$z = i$$

$$z^4 = i^4 = 1$$

$$= RHS$$

2.2 Demoivre's theorum

Statement:-

If "n" is integer positive (or) negative

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

1. $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

Simplify:-

$x = \cos\theta + i\sin\theta$, prove that $\frac{1}{x} = \cos\theta - i\sin\theta$

$x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta.$$

Simplify

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta - i\sin 3\theta)^{-4}}{(\cos\theta - i\sin\theta)(\cos 5\theta + i\sin 5\theta)^3}$$

$$Z = \frac{[(\cos\theta + i\sin\theta)^2]^3 [(\cos\theta + i\sin\theta)^{-3}]^{-4}}{(\cos\theta + i\sin\theta)^{-1} [(\cos\theta + i\sin\theta)^5]^3}$$

$$= (\cos\theta + i\sin\theta)^{6+12+2-15}$$

$$= (\cos\theta + i\sin\theta)^5$$

$$Z = \cos 5\theta + i\sin 5\theta$$

(2) $x = \cos\theta + i\sin\theta$, find the value of $x^3 + \frac{1}{x^3}$

$x = \cos\theta + i\sin\theta$ $\quad x^3 = \cos 3\theta + i\sin 3\theta$

$$\frac{1}{x^3} = \cos 3\theta - i\sin 3\theta$$

$$x^3 + \frac{1}{x^3} = 2\cos 3\theta$$

Roots of a complex Numbers

Steps to find n^{th} root of a complex Numbers

i) write down number in modulus - amplitude form

ii) Add $2k\pi$ with the amplitude

iii) Apply Demoiver's theorem.

iv) put $k = 0, 1, 2 \ldots n-1$

i) Find the value of $1+\omega^2+\omega^4$

$$= 1+\omega^2+\omega^3.\omega$$

$$= 1+\omega^2+\omega = 0$$

Solve

(1) $x^4 - 1 = 0$

$$x^4 = (1)$$

$$x = (1)^{1/4}$$

$$= (\cos 0 + i\sin 0)^{1/4}$$

$$= [\cos 2k\pi + i\sin 2k\pi]^{1/4}$$

$$= \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}$$

$k = 0, 1, 2, 3$

$k=0$ $= \cos\frac{2(0)\pi}{4} + i\sin\frac{2(0)\pi}{4}$

$$= \cos\frac{0\pi}{4} + i\sin\frac{0\pi}{4}$$

$$= \cos 0 + i\sin 0$$

$k=1$ $= \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}$

$k=2$ $= \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}$

$k=3$ $= \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}$

Solve $x^5+1=0$

$$x^5=(-1)$$

$$x=(-1)^{\frac{1}{5}}$$

$$=(\cos\pi+i\sin\pi)^{\frac{1}{5}}$$

$$=(\cos(2k\pi+\pi)+i\sin(2k\pi+\pi))^{\frac{1}{5}}$$

$$=\cos\frac{2k\pi+\pi}{5}+i\sin\frac{2k\pi+\pi}{5}$$

$k=0,1,2,3,4$

$k=0$ $\quad \cos \cancel{\cos}\frac{\pi}{5}+i\sin\frac{\pi}{5}$

$k=1$ $\quad \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$

$k=2$ $\quad \cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5}$

$k=3$ $\quad \cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$

$k=4$ $\quad \cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

Unit III

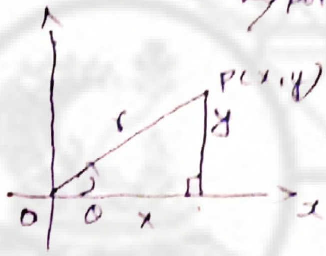
Trigonometry

Compound Angles

Trigonometric ratios

Let $P(x,y)$ be any point on the xoy plane. Draw PM perpendicular to OX and join OP. Let $x\hat{O}P = \theta$ and $OP = r$. Then we have six Trigonometric ratios

i. Sine of angle $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r}$



ii) Cosine of the angle $\cos\theta = \frac{x}{r}$

iii) Tangent of " $\tan\theta = \frac{y}{x}$

iv) Co-secant of " $\csc\theta = \frac{r}{y}$

v) secant " " $\sec\theta = \frac{r}{x}$

vi) cotangent " " $\cot\theta = \frac{x}{y}$.

Definition:-

An angle is defined as being generated by rotating a ray about its fixed point from an initial position to a terminal position.

θ	0°	30°	45°	60°	90°	180°	270°	360
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	1	1	$\sqrt{3}$	∞	0	$-\infty$	0

Compound angles of sin, cos

Formula

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

1. Without using tables, find the value of $\sin 40 \cos 10 - \cos 40^\circ \sin 10$

Solution:-

$$\sin 40^\circ \cos 10 - \cos 40 \sin 10 = \sin(40-10)$$
$$= \sin 30^\circ$$
$$= \frac{1}{2}$$

2. $\sin A = \frac{8}{17}$, $\sin B = \frac{5}{13}$ A and B are acute angles

Show that $\sin(A+B) = \frac{171}{221}$

$$\cos A = \sqrt{1-\sin^2 A}$$
$$= \sqrt{1-\left(\frac{8}{17}\right)^2}$$
$$= \sqrt{1-\frac{64}{289}} = \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}}$$

$$\boxed{\cos A = \frac{15}{17}}$$

III^ly $\boxed{\cos B = \frac{12}{13}}$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$= \frac{8}{17}\times\frac{12}{13} + \frac{15}{17}\times\frac{5}{13}$$

$$\boxed{\sin(A+B) = \frac{171}{221}}$$

Compound angles of tangent

Tangent formula

i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

ii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Part

$\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$ find $\tan(A-B)$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$= \dfrac{\frac{1}{2} - \frac{1}{3}}{1 + \frac{1}{2} \times \frac{1}{3}} = \dfrac{\frac{1}{6}}{\frac{6+1}{6}}$

$$\boxed{\tan(A-B) = \frac{1}{7}}$$

2. $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$ find $\tan(A+B)$ (or) Show that $A+B = 45^{\circ}$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = \dfrac{\frac{3+2}{6}}{1-\frac{1}{6}}$

$\tan(A+B) = 1$

3) $A+B = 45^{\circ}$ P.T $(1+\tan A)(1+\tan B) = 2$ Hence deduce the value $\tan 22\frac{1}{2}^{\circ}$

Solution

$A+B = 45^{\circ}$

$\tan(A+B) = \tan 45^{\circ}$

$\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\tan A + \tan B = 1 - \tan A \tan B$

$\therefore \tan A + \tan B + \tan A \tan B = 1$

$(1+\tan A)(1+\tan B)$

$= 1 + \tan B + \tan A + \tan A \tan B$

$= 1+1$

$= 2$

$(1+\tan A)(1+\tan B) = 2$

$A = B = 22\frac{1}{2}^\circ$

$(1+\tan 22\frac{1}{2}^\circ)(1+\tan 22\frac{1}{2}^\circ) = 2$

$(1+\tan 22\frac{1}{2}^\circ)^2 = 2$

$1+\tan 22\frac{1}{2}^\circ = \sqrt{2}$

$\boxed{\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1}$

Multiple angles

Introduction $\sin 2A$, $\cos 2A$, $\tan 2A$ are called multiple angles. and $\sin A/2$, $\cos A/2$, $\tan A/2$ are called sub-multiple angles

Formula

(ie)

$$\boxed{\begin{aligned} \sin 2A &= 2\sin A \cos A \\ \cos 2A &= 2\cos^2 A - 1 \\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A} \end{aligned}}$$

Prove that $\frac{\sin 2A}{1+\cos 2A} = \tan A$ and hence find the value of $\tan 15^\circ$

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin A}{\cos A} = \tan A$$

Now $\tan 15^\circ = \frac{\sin 2(15^\circ)}{1+\cos 2(15^\circ)} = \frac{\sin 30^\circ}{1+\cos 30^\circ}$

$$= \frac{1/2}{1+\frac{\sqrt{3}}{2}} = \frac{1}{2+\sqrt{3}}$$

① prove that $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2$

$$= \frac{\sin 3A \cos A - \cos 3A \sin A}{\sin A \cos A}$$

$$= \frac{\sin(3A - A)}{\sin A \cos A} = \frac{\sin 2A}{\sin A \cos A}$$

$$= \frac{2 \sin A \cos A}{\sin A \cos A}$$

$$= 2$$

Sum and product formula

Sum formula

$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$

product formula

1. $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$
2. $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$

Prove that

$$\cos(\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4 \sin^2\left(\frac{\alpha-\beta}{2}\right)$$

Solution:-

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$

$$= \left\{-2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + 2\cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)^2$$

$$= 4 \sin^2\left(\frac{\alpha+\beta}{2}\right) \sin^2\left(\frac{\alpha-\beta}{2}\right) + 4\cos^2\left(\frac{\alpha+\beta}{2}\right) \sin^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= 4 \sin^2\left(\frac{\alpha-\beta}{2}\right) \left\{ \sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right) \right.$$

$$= 4 \sin^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= 4 \sin^2\left(\frac{\alpha-\beta}{2}\right)$$

Unit - IV

Inverse Trigonometric Functions

Differential calculus - I

Defn The Quantities $\sin^{-1}x$, $\tan^{-1}x$ ... are called Inverse Trigonometrical functions. That is Inverse circular function $\sin^{-1}x$ is an angle θ. whose cosine is x and so on.

Example:-

$$\sin^{-1}(1/2)$$

$$\text{Let } y = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\sin y = \frac{1}{2}$$

$$= \sin\frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$$

Property - I

i) $\sin^{-1}(\sin x) = x$

ii) $\cos^{-1}(\cos x) = x$

Prove that $$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$$

Solution

$$x = \tan\theta \Rightarrow \theta = \tan^{-1}x$$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

Prove that $$\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

Soln

$$\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\frac{4}{3}-\frac{1}{7}}{1+\left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}\right)$$

$$= \frac{\pi}{4}$$

(1) Prove that $\tan^{-1} y \overset{(ii)}{=} \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ $xy<1$

Proof :-

Let $\theta_1 = \tan^{-1} x \Rightarrow x = \tan\theta_1$

Let $\theta_2 = \tan^{-1} y \Rightarrow y = \tan\theta_2$

$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1 \tan\theta_2}$$

$$\therefore \tan(\theta_1 - \theta_2) = \frac{x-y}{1+xy}$$

$$\theta_1 - \theta_2 = \tan^{-1}\left(\frac{x-y}{1+xy}\right) //$$

(2) Provet that $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3\tan^{-1} x$

solutions

Let $x = \tan\theta$, $\theta = \tan^{-1} x$

$$\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$= 3\tan^{-1} x //$$

Limits

Defn:- The Limit of a function $f(x)$ is that the value to which the function approaches at x approaches to a given value. If $\lim_{x \to a} f(x) = l$, then "l" is called the Limit of the function as "x" approaches "a".

(1) Evaluate $\underset{x \to 0}{Lt} \dfrac{x^2 - 3x + 2}{x + 5}$

$$\underset{x \to 0}{Lt} \frac{x^3 - 3x + 2}{x + 5} = \frac{0^2 - 3(0) + 2}{0 + 5} = \frac{2}{5}$$

(2) Evaluate $\underset{x \to 2}{Lt} \dfrac{x^2 - 3x + 2}{3x + 2}$

$$\underset{x \to 2}{Lt} = \frac{2^2 - 3(2) + 2}{3(2) + 2} = \frac{0}{8} = 0$$

(3) Evaluate $\underset{x \to 0}{Lt} \dfrac{\sin 2x}{3x}$

$$\underset{x \to 0}{Lt} \frac{\sin 2x}{3x} = \frac{1}{3} \underset{x \to 0}{Lt} \frac{\sin 2x}{x}$$

$$= \frac{1}{3}(2)$$

$$= 2/3$$

(4) Evaluate $\underset{x \to 0}{Lt} \dfrac{\sin 10x}{\sin 7x}$

$$\underset{x \to 0}{Lt} \frac{\sin 10x}{\sin 7x} = \underset{x \to 0}{Lt} \frac{\frac{\sin 10x}{x}}{\frac{\sin 7x}{x}}$$

$$= \frac{\underset{x \to 0}{Lt} \frac{\sin 10x}{x}}{\underset{x \to 0}{Lt} \frac{\sin 7x}{x}} = \frac{10}{7}$$

(5) Evaluate $\underset{x \to 3}{Lt} \dfrac{x^5 - 243}{x^3 - 27}$

$$= \underset{x \to 3}{Lt} \frac{\frac{x^5 - 3^5}{x - 3}}{\frac{x^3 - 3^3}{x - 3}}$$

$$= \frac{5 \times 3^{5-1}}{3 \times 3^{3-1}} = \frac{5 \times 3^4}{3 \times 3^2} = 15$$

Unit-V

Differential calculus-II

Differentiation Methods

(1) **Differentiation of function of function (OR) Chain rule**

$y = f(u)$ and $u = g(x)$

then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ is called the function of function (or) chain rule.

$y = f(u)$, $u = g(v)$, $v = h(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}}$$

This rule is called chain rule.

(1) $y = \cos(\log x)$ find $\frac{dy}{dx}$

Solution

$y = \cos u$, where $u = \log x$

$\frac{dy}{du} = -\sin u$, $\frac{du}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \sin u \times \frac{1}{x}$$

$$= -\sin(\log x) \cdot \frac{1}{x}$$

$$= \frac{-\sin \log(x)}{x}$$

Part-B

(1) $y = \sin(\log x^2)$ find $\frac{dy}{dx}$

$y = \sin u$, $u = \log v$, $V = x^2$

$$\frac{dy}{du} = \frac{dy}{du} \times \frac{du}{dv} \times dv/dx$$

$$= \cos u \times \frac{1}{v} \times 2x$$

Differentiation (II)

Formula

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$ (4) $\frac{d}{dx}(\log x) = \frac{1}{x}$

(2) $\frac{d}{dx}(e^x) = e^x$ (5) $\frac{d}{dx}(\cos x) = -\sin x$

(3) $\frac{d}{dx}(k) = 0$

(1)

$y = x^2 \sin x$

$u = x^2, \quad v = \sin x$

$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$

$= x^2 \cos x + \sin x \, 2x$

(2)

$y = e^x \log x \sin x$

$\frac{dy}{dx} = e^x \log x \cos x + \log x \sin x \, e^x + e^x \sin x \frac{1}{x}$

(3)

$y = \frac{\sin x}{e^x}$

$u = \sin x \qquad v = e^x$

$$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$\boxed{\frac{dy}{dx} = \frac{e^x \cos x - \sin x \cdot e^x}{(e^x)^2}}$$

Part - C

① $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, find $\frac{dy}{dx}$

Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$; $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$y = 2\theta = 2\tan^{-1}x$

$\frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2}$ //

② Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$

Diff w.r.t. x

$3x^2 + 3y^2 \frac{dy}{dx} = 3\left(x \cdot \frac{dy}{dx} + y \cdot 1\right)$

$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$

$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$

$3\frac{dy}{dx}(y^2 - x) = 3(y - x^2)$

$\therefore \frac{dy}{dx} = \frac{3(y-x^2)}{3(y^2-x)}$ //

$= \cos u \times \frac{1}{v} \times 2x$

$= \cos(\log x^2) \cdot \frac{1}{x^2} \cdot 2x$

$= \frac{2x \cos(\log x^2)}{x^2} = \frac{2 \cos(\log x^2)}{x}$ //

(1) Find $\frac{dy}{dx}$ if $y = \log\left(\frac{1-\sin x}{1+\sin x}\right)$

Soln

Let $y = \log u$. where $u = \frac{1-\sin x}{1+\sin x}$

$\frac{dy}{du} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{(1+\sin x)\frac{d}{dx}+(1-\sin x) - (1-\sin x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$

$= \frac{-\cos x - \sin x \cos x - \cos x + \cos x \cdot \sin x}{(1+\sin x)^2}$

$\frac{dy}{dx} = \frac{-2\cos x}{(1+\sin x)^2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{-2\cos x}{\left(\frac{1-\sin x}{1+\sin x}\right)(1+\sin x)^2}$

$$\boxed{\frac{dy}{dx} = \frac{-2\cos x}{(1-\sin x)(1+\sin x)}}$$

//

. Find $\frac{dy}{dx}$ if $x^3+y^3 = 3xy$

Diff. w.r.t. x

$$3x^2 + 3y^2\frac{dy}{dx} = 3\left(x \cdot \frac{dy}{dx} + y \cdot 1\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$3y^2 \cdot \frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$3\frac{dy}{dx}(y^2 - x) = 3(y - x^2)$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{3(y-x^2)}{3(y^2-x)}}$$

Part - C

(1) ~~y = cosx~~ $y = x\cos x$, prove that $x^2y_2 - 2xy_1 + (x^2+2)y = 0$

Solution

$$y = x\cos x$$

$$y_1 = x(-\sin x) + \cos x \cdot 1$$

$$= -x\sin x + \cos x$$

$$y_2 = -\{x \cdot \cos x + \sin x \cdot 1\} - \sin x$$

$$= -x\cos x - 2\sin x$$

LHS $x^2y_2 - 2xy_1 + (x^2+2)y$

$$= x^2(-x\cos x - 2\sin x) - 2x(-x\sin x + \cos x) + \ldots$$

$$-x^3\cos x - 2x^2\sin x + 2x^2\sin x - 2x\cos x + x^3\cos x +$$

$$2x\cos x$$

$$= 0$$

$$= RHS$$

(2) $xy = ae^{x} + be^{-x}$ prove that $xy_2 + 2y_1 = xy$.

$xy = ae^{x} + be^{-x}$

Diff w.r.t x on both sides

$x \cdot y_1 + y \cdot 1 = ae^{x} + b[e^{-x} \cdot (-1)]$

$= ae^{x} - be^{-x}$

$xy_2 + y_1 \cdot 1 + y_1 = ae^{x} - b[e^{-x}(-1)]$

$= ae^{x} + be^{-x}$

$= by$

$\boxed{xy_2 + 2y_1 = xy}$

5.3 Partial differentiation

Worked Examples

part

(1) $u = x^3 + 5x^2y + y^3$ find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$.

$u = x^3 + 5x^2y + y^3$

$\frac{\partial u}{\partial x} = 3x^2 + 5y(2x) + 0 = 3x^2 + 10xy$

$\frac{\partial u}{\partial y} = 0 + 5x^2(1) + 3y^2 = 5x^2 + 3y^2$

(1) If $u = x^3 - 2x^2y + 3xy^2 + y^3$ find $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$

$$u = x^3 - 2x^2y + 3xy^2 + y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 2y(2x) + 3y^2(1) + 0$$

$$= 3x^2 - 4xy + 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x - 4y$$

$$\frac{\partial u}{\partial y} = -2x^2 + 6xy + 6y$$

$$\frac{\partial^2 u}{\partial y^2} = 6x + 6$$

(2) If $u = \log(x^2+y^2)$ find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\boxed{x \cdot \frac{\partial u}{\partial x} = \frac{2x^2}{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\boxed{y \cdot \frac{\partial u}{\partial y} = \frac{2y^2}{x^2+y^2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2+2y^2}{x^2+y^2}$$

$$= 2\left(\frac{x^2+y^2}{x^2+y^2}\right) = 2$$