Unit - I

1.1. Analytical Geometry - II.

Circle:

Defn:

A circle is the locus of a point which moves in a plane such way that its distance from a fixed point is constant.

The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Fixed point = C (centre)

Constant Distance = CP = r (radius)

Equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2.$$

Centre = (h, k) and Radius = r.

Diameter form of a circle:

The eqn of the circle described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.

$$\therefore (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0.$$

General Equations of a circle:

$$x^2+y^2+2gx+2fy+c=0$$

centre $= (-g, -f)$

radius $r = \sqrt{g^2+f^2-c}$

Family of circles:

concentric circles:

Two or more circles having the same centre but different radii are called concentric circles.

Ex:

$$x^2+y^2+8x-12y+13=0$$

$$x^2+y^2+8x-12y+25=0$$

Here the circles differ only by constant term

i) $c_1 \neq c_2$

orthogonal circles

(condition only)

Two circles are said to be orthogonal if the tangents at their point of intersection are at right angles.

Condition for two circles cut orthogonally:

$$2g_1g_2+2f_1f_2=c_1+c_2.$$

· Contact of circles:

Two circles may touch each other either internally or externally.

(1) Condition for two circles to touch Externally:

$$\boxed{d = C_1C_2 = r_1 + r_2}$$

Point of contact:

$$\left[\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right]$$

(2) Condition for two circles to touch Internally:

(i) $$\boxed{d = C_1C_2 = r_1 - r_2 \text{ or } r_2 - r_1}$$

Point of contact:

$$\left[\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right]$$

Part - A

(1) Find the equation of the circle, whose centre (0,0) and radius $\sqrt{5}$ units.

Soln

Formula: $x^2 + y^2 = r^2$

$$x^2 + y^2 = (\sqrt{5})^2$$

$$x^2 + y^2 = 5$$

② Find the centre and radius of the circle $x^2+y^2=18$.

Soln

Formula : $x^2+y^2=r^2$

Given $x^2+y^2=18 \Rightarrow x^2+y^2=(\sqrt{18})^2$

centre $=(0,0)$ and Radius $r=\sqrt{18}$ units.

③ Show that the circles $x^2+y^2-4x+2y+5=0$ and $x^2+y^2-4x+2y-8=0$ are concentric circles.

Soln:

Given $x^2+y^2-4x+2y+5=0$

$x^2+y^2-4x+2y-8=0$

Here $c_1=5$ and $c_2=-8$

$\therefore c_1 \neq c_2$

$\therefore$ Given circles are concentric.

Part-B

④ Find the equation of the circle whose centre is $(-5,7)$ and radius 3 units.

Soln

$(x-h)^2+(y-k)^2=r^2$

Here $(h,k)=(-5,7)$ and $r=3$

$\therefore (x+5)^2+(y-7)^2=3^2$

$x^2+25+10x+y^2+49-14y=9$

$x^2+y^2+10x-14y+65=0$

5) Find the equation of the circle with centre $(-3,-4)$ passing through the point $(2,2)$.

Soln

centre $C = (-3,-4)$

point $(2,2)$

Equation of circle is $(x-h)^2+(y-k)^2=r^2$

Put $(h,k) = (-3,-4)$

$$(x+3)^2+(y+4)^2=r^2 \longrightarrow ①$$

$(x,y) = (2,2)$

$$(2+3)^2+(2+4)^2=r^2$$

$$r^2 = 25+36$$

$$r^2 = 61$$

$$① \Rightarrow (x+3)^2+(y+4)^2 = 61.$$

6) Find the centre and radius of the circle $(x-4)^2+(y+6)^2=4$.

Soln

We know that $(x-h)^2+(y-k)^2=r^2$.

$$(x-4)^2+(y+6)^2=2^2$$

$\therefore$ centre $= (4,-6)$ and Radius $r = 2$ units.

7) Prove that circles $x^2+y^2-4x+6y+4=0$ and $x^2+y^2+2x+4y+4=0$ cut orthogonally.

Soln

$$x^2+y^2-4x+6y+4=0 \rightarrow (1)$$

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \rightarrow (2)$$

(1) & (2)

$$\Rightarrow 2g_1=-4 \quad \boxed{g_1=-2} \quad \Big| \quad 2f_1=6 \quad \boxed{f_1=3} \quad \Big| \quad c_1=4$$

$$x^2+y^2+2x+4y+4=0 \rightarrow (3)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \rightarrow (4)$$

(3) & (4)

$$\Rightarrow 2g_2=2 \quad \boxed{g_2=1} \quad \Big| \quad 2f_2=4 \quad \boxed{f_2=2} \quad \Big| \quad c_2=4$$

Formula:

$$2g_1g_2+2f_1f_2=c_1+c_2$$

$$2(-2)(1)+2(3)(2)=4+4$$

$$-4+12=8$$

$$8=8$$

$\therefore$ circles cut each other orthogonally.

Part - C

8. Find the equation of the circle two of whose diameters are $x+y=6$ and $x+2y=4$ and whose radius is 10 units.

Soln.

$$x+y=6 \rightarrow (1)$$

$$\underset{(-)\ \ (-)\ \ \ \ (-)}{x+2y=4} \rightarrow (2)$$

$$(1)-(2) \Rightarrow \quad -y = 2 \Rightarrow y=-2$$

Put $y=-2$ in (1) $\quad x+y=6 \Rightarrow x-2=6$

$$\Rightarrow x=8$$

$\therefore$ centre is $(8,-2)$

Equation of circle is $(x-h)^2+(y-k)^2=r^2$

$$(x-8)^2+(y+2)^2=10^2$$

$$x^2-16x+64+y^2+4y+4=100$$

$$x^2+y^2-16x+4y-32=0.$$

9. Find the equation of the circle passing through the point $(-7,1)$ and having centre at $(-4,-3)$.

Soln

Let $C(-4,-3)$ be the centre and $P(-7,1)$ be the point on the circle.

The distance between centre C and the point P on the circle is

$$CP = r = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{(-7+4)^2+(1+3)^2}$$

$$= \sqrt{(-3)^2+(4)^2}$$

$$= \sqrt{9+16} = \sqrt{25} = 5$$

Equation of circle is $(x-h)^2+(y-k)^2=r^2$

$$(x+4)^2+(y+3)^2=5^2$$

$$x^2+8x+16+y^2+6y+9=25$$

$x^2+y^2+8x+6y=0$ is the required equation of the circle.

10. If the diameter of the circle is a line joining the points $A(2,-1)$ and $B(-4,5)$, find the ~~equation~~ equation of the circle. Also find the centre and radius of the circle.

Soln

Equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

$(x-2)(x+4)+(y+1)(y-5)=0$

$x^2-2x+4x-8+y^2-5y+y-5=0$

$x^2+2x-8+y^2-4y-5=0$

$x^2+y^2+2x-4y-13=0.$

is in the form of

$x^2+y^2+2gx+2fy+c=0.$

Centre $=(-g,-f)=(-1,2)$

Radius $r=\sqrt{g^2+f^2-c}=\sqrt{(1)^2+(-2)^2-(-13)}$

$=\sqrt{1+4+13}=\sqrt{18}=3\sqrt{2}$ units.

11. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ touch other. Find also the point of contact.

Soln

$x^2+y^2-4x+6y+8=0$

$2g=-4$ $g=-2$	$2f=6$ $f=3$	$c=8$

Centre $C_1=(-g,-f)=(2,-3)$

Radius $r_1=\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+3^2-8}=\sqrt{5}$

$x^2+y^2-10x-6y+14=0$

$2g=-10$ | $2f=-6$ | $c=14$
$g=-5$ | $f=-3$ |

centre $C_2=(-g,-f)=(5,3)$

Radius $r_2=\sqrt{g^2+f^2-c}$

$=\sqrt{(-5)^2+(-3)^2-14}$

$=2\sqrt{5}$

Distance between the centres C_1 and C_2 is

$C_1C_2=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(5-2)^2+(3+3)^2}$

$=\sqrt{3^2+6^2}=\sqrt{9+36}$

$=\sqrt{45}=3\sqrt{5}$

$r_1+r_2=\sqrt{5}+2\sqrt{5}=3\sqrt{5}=C_1C_2$

$\therefore$ circles touch other externally.

Point of contact

$$\left(\frac{r_1x_2+r_2x_1}{r_1+r_2},\ \frac{r_1y_2+r_2y_1}{r_1+r_2}\right)$$

$$= \left(\frac{2(-1)+3(2)}{2+3}, \frac{2(-1)+3(3)}{2+3} \right)$$

$r_1 : r_2 = 2 : 3$

$\cancel{x_1 + x_2} =$

$(x_1, y_1) = (2, 3)$

$(x_2, y_2) = (-1, -1)$

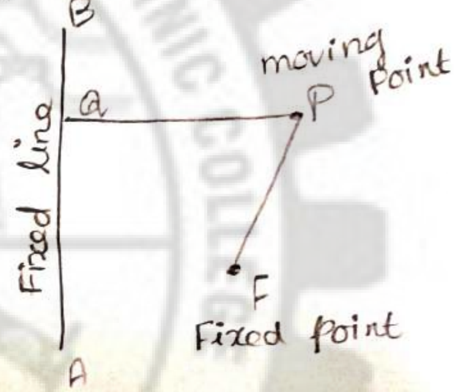
$$= \left(\frac{-2+6}{5}, \frac{-2+9}{5} \right)$$

$$= \left(\frac{4}{5}, \frac{7}{5} \right)$$

1.2. CONICS

Defn:

A conic is the locus of a point which moves in a plane such that ratio of its distance from a fixed point and its distance from a fixed line is always constant.



In Fig AB is a fixed line. F is the fixed point and P is the moving point and PQ is the distance of P from fixed line.

By defn. of conic $\frac{FP}{PQ}$ = constant = e say

Equation of conic is $\frac{FP}{PQ} = e$.

The fixed point is called Focus, fixed ~~line~~ line is called directrix and the constant e is called eccentricity.

Equation of conic:

$$(x-x_1)^2+(y-y_1)^2 = e^2 \frac{(lx+my+n)^2}{l^2+m^2}$$

classification of conics:

a) Parabola:

General Equation of conic:

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

classification of conic:

a) Parabola:

$h^2-ab=0$ and $e=1$.

b) Ellipse:

$h^2-ab<0$ and $e<1$.

c) Hyperbola:

$h^2-ab>0$ and $e>1$.

Pair of lines:

The equation of two straight lines not passing through origin are $ax+by+c=0$ and $lx+my+n=0$.

The condition for a quadratic equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent pair of lines (neither conic nor circle) is $abc+2fgh-af^2-bg^2-ch^2=0$ which in determinant form is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Examples

Part - A

① Show that the equation $9x^2-12xy+4y^2+2x-y+1=0$ represents a parabola.

Solution:

The eqn $9x^2-12xy+4y^2+2x-y+1=0$

is in the form of $ax^2+2hxy+by^2+2gx+2fy+c=0$

where $a=9$, $2h=-12$, $b=4$

$$h^2-ab = (-6)^2-(9)(4)$$

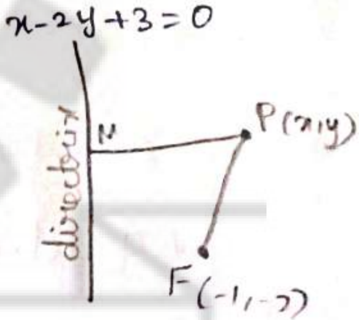
$$= 36-36 = 0$$

$\therefore$ The given equation represents a parabola.

Part-C

(2) Find the equation of parabola whose focus is $(-1,-2)$ and directrix is $x-2y+3=0$.

Solution:



Let $F(-1,-2)$ be the focus and $x-2y+3=0$ is the directrix.

Let $P(x,y)$ be any point on the parabola such that $FP = PM$

$$FP = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$= \sqrt{(x+1)^2+(y+2)^2}$$

$$PM = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right| = \left|\frac{x-2y+3}{\sqrt{1^2+(-2)^2}}\right|$$

$$= \left|\frac{x-2y+3}{\sqrt{5}}\right|$$

$FP = PM$

$$\Rightarrow \sqrt{(x+1)^2+(y+2)^2} = \left|\frac{x-2y+3}{\sqrt{5}}\right|$$

$$(x+1)^2+(y+2)^2 = \frac{(x-2y+3)^2}{5}$$

$$5\left[x^2+2x+1+y^2+4y+4\right] = \left[x^2+4y^2+9-4xy-12y+6x\right]$$

$$4x^2+y^2+4xy+4x+32y+16=0$$

$4x^2+4xy+y^2+4x+32y+16=0$ is the required equation of the Parabola

(3) Show that the equation $21x^2 - xy - 10y^2 + 5x + 13y - 4 = 0$ represents pair of lines.

<u>Soln:</u>

$21x^2 - xy - 10y^2 + 5x + 13y - 4 = 0$ is in the form of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

condition for this equation to represent pair of lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

(i.e)
$$\begin{vmatrix} 42 & -1 & 5 \\ -1 & -20 & 13 \\ 5 & 13 & -8 \end{vmatrix}$$

$$= 42(160-169) + 1(8-65) + 5(-13+100)$$

$$= 42(-9) + 1(-57) + 5(87)$$

$$= -378 - 57 + 435$$

$$= 0$$

$\therefore$ The given equation represents pair of straight lines.

Unit - II

Vector Algebra.

2.1. vector:

Defn:

The physical quantity which has both magnitude and direction is called a vector.

Ex:

The quantities, displacement, velocity, acceleration, force, momentum,

Defn of scalar:

The physical quantity which has only magnitude not related to any fixed direction is called a scalar quantity.

Ex:

the quantities, length, mass, time, area volume, density, speed, temperature

Modulus (or) Magnitude:

The modulus (or) magnitude (or) length of a vector $\vec{a} = \vec{AB}$ is a positive real number which is a measure of its length and is denoted by $|\vec{a}| = a$.

Types of vectors:

Zero (or) Null vector:

A vector whose initial and final points are coincident is called a null vector.

Unit vector:

A vector, whose modulus is one unit is called a unit vector. $|\hat{a}| = 1$.

Unit vector along a vector $\vec{x}$ is

$$\boxed{\hat{x} = \frac{\vec{x}}{|\vec{x}|}}$$

Part-A

(1) Find the sum of the two vectors $\vec{a}-2\vec{b}+3\vec{c}$ and $-2\vec{a}+3\vec{b}-\vec{c}$.

Soln

$$\vec{x}+\vec{y} = \vec{a}-2\vec{b}+3\vec{c} + (-2\vec{a}+3\vec{b}-\vec{c})$$

$$= -\vec{a}+\vec{b}+2\vec{c}$$

(2) If $\vec{a} = 5\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b} = -3\vec{i}-2\vec{j}+5\vec{k}$ find $3\vec{a}+2\vec{b}$.

Soln:

$$3\vec{a}+2\vec{b} = 3(5\vec{i}+2\vec{j}-3\vec{k})+2(-3\vec{i}-2\vec{j}+5\vec{k})$$

$$= 15\vec{i}+6\vec{j}-9\vec{k}-6\vec{i}-4\vec{j}+10\vec{k}$$

$$= 9\vec{i}+2\vec{j}+\vec{k}.$$

3). Find the modulus of the vector $\vec{i} + 5\vec{j} - 7\vec{k}$.

Soln:

Let $\vec{a} = \vec{i} + 5\vec{j} - 7\vec{k}$

$$|\vec{a}| = \sqrt{1^2 + 5^2 + (-7)^2}$$

$$= \sqrt{1 + 25 + 49}$$

$$= \sqrt{75} = 5\sqrt{3}.$$

4) Find the unit vector along the direction of the vector $7\vec{i} + 5\vec{j} - 3\vec{k}$.

Soln

Let $\vec{a} = 7\vec{i} + 5\vec{j} - 3\vec{k}$

$$\therefore |\vec{a}| = \sqrt{7^2 + 5^2 + (-3)^2}$$

$$= \sqrt{49 + 25 + 9} = \sqrt{83}.$$

Unit vector along the direction of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

$$= \frac{7\vec{i} + 5\vec{j} - 3\vec{k}}{\sqrt{83}}$$

Part-B

5) Show that $2\vec{i}-3\vec{j}+5\vec{k}$ and $-6\vec{i}+9\vec{j}-15\vec{k}$ are parallel vectors.

Soln:

$$\vec{a}=2\vec{i}-3\vec{j}+5\vec{k} \text{ and } \vec{b}=-6\vec{i}+9\vec{j}-15\vec{k}$$

$$\vec{b}=-3(2\vec{i}-3\vec{j}+5\vec{k})$$

$$\vec{b}=-3\vec{a}.$$

$\therefore$ $\vec{a}$ & $\vec{b}$ are Parallel vectors.

6) Find the direction cosines of the vector $\vec{i}+2\vec{j}-3\vec{k}$.

Soln

Let $\vec{a}=\vec{i}+2\vec{j}-3\vec{k}$

$$|\vec{a}|=\sqrt{1^2+2^2+(-3)^2}$$

$$=\sqrt{1+4+9}=\sqrt{14}$$

Direction cosines are $\left(\frac{x}{|\vec{r}|},\frac{y}{|\vec{r}|},\frac{z}{|\vec{r}|}\right)$

ie) $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{-3}{\sqrt{14}}\right)$ are the direction cosines.

Part - C

(7) The position vectors of the points $4\vec{i}+2\vec{j}+3\vec{k}$, $2\vec{i}+3\vec{j}+4\vec{k}$ and $3\vec{i}+4\vec{j}+2\vec{k}$. Prove that these points form an equilaternal triangle.

Soln:

Let $\vec{OA} = 2\vec{i}+3\vec{j}+4\vec{k}$

$\vec{OB} = 4\vec{i}+2\vec{j}+3\vec{k}$

$\vec{OC} = 3\vec{i}+4\vec{j}+2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (3\vec{i}+4\vec{j}+2\vec{k}) - (2\vec{i}+3\vec{j}+4\vec{k})$

$= \vec{i}+\vec{j}-2\vec{k}$.

$|\vec{AB}| = \sqrt{1^2+1^2+(-2)^2}$

$= \sqrt{6}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (4\vec{i}+2\vec{j}+3\vec{k}) - (3\vec{i}+4\vec{j}+2\vec{k})$

$= \vec{i}-2\vec{j}+\vec{k}$

$|\vec{BC}| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$

$$\vec{AC} = \vec{OC} - \vec{OA}$$

$$= (4\vec{i} + 2\vec{j} + 3\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$$

$$= 2\vec{i} - \vec{j} - \vec{k}.$$

$$|\vec{AC}| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

$\therefore AB = BC = AC = \sqrt{6}$

$\therefore \Delta ABC$ is an equilateral triangle.

8) Show that the position vectors $2\vec{i} - \vec{j} + 3\vec{k}$, $3\vec{i} - 5\vec{j} + \vec{k}$ and $-\vec{i} + 11\vec{j} + 9\vec{k}$ are collinear.

Soln:

$\vec{OA} = 2\vec{i} - \vec{j} + 3\vec{k}$,

$\vec{OB} = 3\vec{i} - 5\vec{j} + \vec{k}$

$\vec{OC} = -\vec{i} + 11\vec{j} + 9\vec{k}$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= (3\vec{i} - 5\vec{j} + \vec{k}) - (2\vec{i} - \vec{j} + 3\vec{k})$$

$$= \vec{i} - 4\vec{j} - 2\vec{k}$$

$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$= (-\vec{i} + 11\vec{j} + 9\vec{k}) - (3\vec{i} - 5\vec{j} + \vec{k})$$

$$= -4\vec{i} + 16\vec{j} + 8\vec{k}$$

$$= -4(\vec{i} - 4\vec{j} - 2\vec{k})$$

$\therefore \vec{BC} = -4\,\vec{AB}$

$\therefore \vec{AB}$ and $\vec{BC}$ are parallel vectors.

$\therefore$ A, B and C are collinear.

2.2. Product of two vectors.

① What are the value of $\vec{i}\cdot\vec{j}$ and $\vec{k}\cdot\vec{k}$.

Soln

$\vec{i}\cdot\vec{j}=0$ and $\vec{k}\cdot\vec{k}=1$.

$\vec{k}\cdot\vec{i}=0$.

② Find the dot product of the vectors $5\vec{i}-3\vec{k}$ and $4\vec{i}+7\vec{j}+5\vec{k}$.

Soln

Let $\vec{a}=5\vec{i}-3\vec{k}$.

$\vec{b}=4\vec{i}+7\vec{j}+5\vec{k}$.

$\vec{a}\cdot\vec{b}=(5\vec{i}+0\vec{j}-3\vec{k})\cdot(4\vec{i}+7\vec{j}+5\vec{k})$

$=20+0-15=5$

③ Show that the vectors $\vec{i}-3\vec{j}+5\vec{k}$ and $-2\vec{i}+6\vec{j}+4\vec{k}$ are mutually perpendicular.

Soln:

Let $\vec{a}=\vec{i}-3\vec{j}+5\vec{k}$

$\vec{b}=-2\vec{i}+6\vec{j}+4\vec{k}$

$\vec{a}\cdot\vec{b}=(\vec{i}-3\vec{j}+5\vec{k})\cdot(-2\vec{i}+6\vec{j}+4\vec{k})$

$=-2-18+20=0$

$\vec{a}\cdot\vec{b}=0$

$\therefore \vec{a}$ and $\vec{b}$ are mutually perpendicular.

4) If $|\vec{a}+\vec{b}| = 30$, $|\vec{a}-\vec{b}| = 10$ and $|\vec{b}| = 3$ find $|\vec{a}|$.

Soln:

WKT $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

$$30^2 + 10^2 = 2\left[|\vec{a}|^2 + 3^2\right]$$

$$900 + 100 = 2|\vec{a}|^2 + 18$$

$$|\vec{a}|^2 = 491$$

$$|\vec{a}| = \sqrt{491}.$$

5) Find the angle between the vectors $3\vec{i} - 2\vec{j} + 5\vec{k}$ and $2\vec{i} + \vec{j} + 2\vec{k}$.

Soln

Let $\vec{a} = 3\vec{i} - 2\vec{j} + 5\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k}$.

$$|\vec{a}| = \sqrt{3^2 + (-2)^2 + 5^2} = \sqrt{9+4+25} = \sqrt{38}.$$

$$|\vec{b}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\vec{a}\cdot\vec{b} = (3)(2) + (-2)(1) + 5(2)$$

$$= 6 - 2 + 10 = 14$$

Angle between $\vec{a}$ and $\vec{b}$ is $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

$$= \cos^{-1}\left(\frac{14}{3\sqrt{38}}\right)$$

6) S.T the vectors $\vec{i}+2\vec{j}-1\vec{k}$, $\vec{i}+\vec{j}-3\vec{k}$ and $7\vec{i}-4\vec{j}+\vec{k}$ are mutually perpendicular.

Soln

Let $\vec{a} = \vec{i}+2\vec{j}+\vec{k}$

$\vec{b} = \vec{i}+\vec{j}-3\vec{k}$

$\vec{c} = 7\vec{i}-4\vec{j}+\vec{k}$

$\vec{a}\cdot\vec{b} = (\vec{i}+2\vec{j}+\vec{k})\cdot(\vec{i}+\vec{j}-3\vec{k})$

$= 1+2-3 = 0$

$\Rightarrow \vec{a}$ & $\vec{b}$ are perpendicular.

$\vec{b}\cdot\vec{c} = (\vec{i}+\vec{j}-3\vec{k})\cdot(7\vec{i}-4\vec{j}+\vec{k})$

$= (1)(7)+(1)(-4)+(-3)(1)$

$= 7-4-3 = 0$

$\Rightarrow \vec{b}$ & $\vec{c}$ are perpendicular.

$\vec{c}\cdot\vec{a} = (7\vec{i}-4\vec{j}+\vec{k})\cdot(\vec{i}+2\vec{j}+\vec{k})$

$= 7(1)+(-4)(2)+(1)(1)$

$= 7-8+1 = 0$

$\Rightarrow \vec{c}$ & $\vec{a}$ are perpendicular.

$\therefore \vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular.

Vector Product or Cross Product of two vectors.

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	0	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	0	$-\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	0

① Find the unit vector perpendicular to both the vectors $\vec{i}-\vec{j}+2\vec{k}$ and $2\vec{i}+3\vec{j}-\vec{k}$. Also calculate the sine of the angle between the two vectors.

Soln

Let $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$

$\vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$= \vec{i}(1-6) - \vec{j}(-1-4) + \vec{k}(3+2)$

$= -5\vec{i} + 5\vec{j} + 5\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + 5^2 + 5^2} = \sqrt{25+25+25}$

$= \sqrt{75} = 5\sqrt{3}$.

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$= \frac{5(-\vec{i} + \vec{j} + \vec{k})}{5\sqrt{3}} = \frac{-\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

$$|\vec{a}| = \sqrt{1+1+4} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{4+9+1} = \sqrt{14}$$

$$\therefore \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} = \frac{5\sqrt{3}}{\sqrt{6}\,\sqrt{14}}$$

$$= \frac{5\sqrt{3}}{\sqrt{2\times 3}\,\sqrt{7\times 2}} = \frac{5}{2\sqrt{7}}$$

Application of scalar and vector product.

(1). Find the work done by the force $2\vec{i}+\vec{j}+1\vec{k}$ acting on the particle, if the particle is displaced from $4\vec{i}+\vec{j}+3\vec{k}$ to the point $5\vec{i}+4\vec{j}+2\vec{k}$.

Soln.

$$\vec{F} = 2\vec{i}+\vec{j}+\vec{k}$$

$$\vec{OA} = 4\vec{i}+\vec{j}+3\vec{k}$$

$$\vec{OB} = 5\vec{i}+4\vec{j}+2\vec{k}$$

displacement $\vec{d} = \vec{AB} = \vec{OB} - \vec{OA}$

$$= (5\vec{i}+4\vec{j}+2\vec{k}) - (4\vec{i}+\vec{j}+3\vec{k})$$

$$= \vec{i}+3\vec{j}-\vec{k}$$

$\therefore$ Workdone $= \vec{F}\cdot\vec{d}$

$$= (2\vec{i}+\vec{j}+\vec{k}) \cdot (\vec{i}+3\vec{j}-\vec{k})$$

$$= 2(1) + (1)(3) + (1)(-1)$$

$$= 2+3-1 = 4 \text{ units.}$$

3) The forces $2\vec{i}-5\vec{j}+6\vec{k}$, $-\vec{i}+2\vec{j}-\vec{k}$ and $2\vec{i}+7\vec{j}$ act on a particle and displaced it from the point $4\vec{i}-3\vec{j}-2\vec{k}$ to the point $6\vec{i}+\vec{j}-3\vec{k}$. Find the total work done by the force.

<u>soln</u>

$$\vec{F_1} = 2\vec{i}-5\vec{j}+6\vec{k}$$

$$\vec{F_2} = -\vec{i}+2\vec{j}-\vec{k}$$

$$\vec{F_3} = 2\vec{i}+7\vec{j}$$

$$\vec{F} = \vec{F_1}+\vec{F_2}+\vec{F_3}$$

$$= (2\vec{i}-5\vec{j}+6\vec{k}) + (-\vec{i}+2\vec{j}-\vec{k}) + (2\vec{i}+7\vec{j})$$

$$= 3\vec{i}+4\vec{j}+5\vec{k}$$

Let $\vec{OA} = 4\vec{i}-3\vec{j}-2\vec{k}$, $\vec{OB} = 6\vec{i}+\vec{j}-3\vec{k}$

displacement $\vec{AB} = \vec{OB}-\vec{OA}$

$$= (6\vec{i}+\vec{j}-3\vec{k}) - (4\vec{i}-3\vec{j}-2\vec{k})$$

$$\vec{d} = 2\vec{i}+4\vec{j}-\vec{k}$$

Total work done $= \vec{F}\cdot\vec{d}$

$$= (3\vec{i}+4\vec{j}+5\vec{k})\cdot(2\vec{i}+4\vec{j}-\vec{k})$$

$$= 3(2)+4(4)+5(-1)$$

$$= 6+16-5 = 17 \text{ units.}$$

UNIT- III

INTEGRAL CALCULUS-I

3.1 INTEGRATION USING DECOMPOSITION METHOD

Definition :-

If the derivative of a Function $F(x)$ w.r to 'x' is $f(x)$ then we say that integral of $f(x)$ w.r.t 'x' is $F(x)$.

(i.e.,) If $\frac{d}{dx}[F(x)] = f(x)$ then $\int f(x)dx = F(x)$.

Differentiation	Integration
1. $\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2. $\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
3. $\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log x + c$
4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}} dx = \sqrt{x+c}$ or $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$
5. $\frac{d}{dx}(x) = 1$	$\int 1 dx = x + c$
6. $\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
7. $\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin dx = -\cos + c$

8. $\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + c$
9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
10. $\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + c$
11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

PART-A

1. Evaluate $\int (x^2 - 3x + 2)dx$

Solution:-

$$\int (x^2 - 3x + 2)dx = \int x^2\, dx - 3\int x\, dx + 2\int dx$$

$$= \left[\frac{x^3}{3}\right] - 3\left[\frac{x^2}{2}\right] + 2x + c$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$$

2. Evaluate : $\int (x^2 + \sec^2 x)dx$

Solution:

$$\int (x^2 + \sec^2 x)dx = \frac{x^3}{3} + \tan x + c$$

PART-B AND C

1. Evaluate : $\int (2x^4 - 8x^3 + 11x^2 + 7)\, dx$

Solution:-

$$\int (2x^4 - 8x^3 + 11x^2 + 7)dx = 2\left[\frac{x^5}{5}\right] - 8\left[\frac{x^4}{4}\right] + 11\left[\frac{x^3}{3}\right] + 7x + C$$

$$= \frac{2x^5}{5} - 2x^4 + \frac{11x^3}{3} + 7x + C$$

2. Evaluate : $\int (x^2 + x - 1)(x^2 - x + 1)dx$

Solution:

$$\int (x^2+x-1)(x^2-x+1)dx = \int (x^4 - x^3 + x^2 + x^3 - x^2 + x - x^2 + x - 1)\, dx$$

$$= \int (x^4 - x^2 + 2x - 1)dx$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + \frac{2x^2}{2} - x + C$$

$$= \frac{x^5}{5} - \frac{x^3}{3} - x^2 - x + C$$

1. Evaluate : $\int \left(e^{2x} + \frac{1}{2x-3}\right] dx$

Solution:.

$$\int \left(e^{2x} + \frac{1}{2x-3}\right) dx = \frac{e^{2x}}{2} + \frac{\log(2x-3)}{2} + C$$

2. Evaluate: $\int \sec^2 5x\, dx$

solution:

$$\int \sec^2 5x\, dx = \frac{1}{5}(\tan 5x) + C$$

1. Evaluate : $\int \sin^3 x\, dx$

Solution :

$$\int \sin^3 x\, dx = \int\left[\frac{3\sin x - \sin 3x}{4}\right] dx$$

$$\left[\sin 3x = 3\sin x - 4\sin^3 x\right]$$

$$\left[\sin^3 x = \frac{3\sin x - \sin 3x}{4}\right]$$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)\, dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$$

2. Evaluate : $\int \cos^3 x\, dx$

Solution :-

$$\int \cos^3 x\, dx = \int\left[\frac{3\cos x + \cos 3x}{4}\right] dx$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\left[\cos^3 x = \frac{3\cos x + \cos 3x}{4}\right]$$

$$= \frac{1}{4}\int (3\cos x + \cos 3x)\, dx$$

<u>Decomposition by partial fraction method</u> :-

1. Evaluate : $\int \frac{x+1}{x^2+5x+6}\, dx$

Let $I = \int \frac{x+1}{x^2+5x+6}\, dx = \int \frac{x+1}{(x+3)(x+2)}\, dx$

Let $\frac{x+1}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$

$$\frac{x+1}{(x+3)(x+2)} = \frac{A(x+2) + B(x+3)}{(x+3)(x+2)}$$

$$\Rightarrow x+1 = A(x+2) + B(x+3)$$

Put $x = -2$, then $-1 = A(0) + B(-1)$

$\therefore B = -1$

Put $x=-3$, then $-2 = A(-1)+B(0)$

$\therefore A = 2$

$$\therefore \frac{x+1}{(x+3)(x+2)} = \frac{2}{x+3} - \frac{1}{x+2}$$

$$\therefore I = \int \frac{x+1}{x^2+5x+6}\,dx = \int\left(\frac{2}{x+3} - \frac{1}{x+2}\right)dx$$

$$= 2\log(x+3) - \log(x+2) + c$$

2. Evaluate :-

$$\int \frac{x+3}{(x+2)^2(x+1)}\,dx$$

Solution :-

$$\text{Let } I = \int \frac{x+3}{(x+2)^2(x+1)}\,dx$$

$$\text{Let } \frac{x+3}{(x+2)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\frac{x+3}{(x+2)^2(x+1)} = \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}$$

$$\Rightarrow x+3 = A(x+2^2) + B(x+1)(x+2) + C(x+1)$$

Put $x = -1$ then

$2 = A(1)^2 + B(0)(2) + C(0)$

$2 = A$

Put $x = -2$ then

$1 = A(0)^2 + B(-1)(0) + C(-1)$

$1 = -C$

$\therefore C = -1$

Put $x = 0$, then

$$3 = A(2)^2 + B(1)(2) + C(1)$$

$$3 = 4A + 2B + C$$

$$3 = 4(2) + 2B - 1$$

$$3 = 8 + 2B - 1 \rightarrow 2B = -4$$

$$\therefore B = -2$$

$$\frac{x+3}{(x+2)^2(x+1)} = \frac{2}{x+1} - \frac{2}{x+2} - \frac{1}{(x+2)^2}$$

$$\int \frac{x+3}{(x+2)^2(x+1)}\,dx = \int \left[\frac{2}{x+1} - \frac{2}{x+2} - \frac{1}{(x+2)^2}\right] dx$$

$$= \int \left[\frac{2}{x+1} - \frac{2}{x+2} - (x+2)^{-2}\right] dx$$

$$= 2\log(x+1) - 2\log(x+2) - \frac{(x+2)^{-1}}{-1} + c$$

$$= 2\log(x+1) - 2\log(x+2) + \frac{1}{x+2} + c$$

3.2 INTEGRATION BY SUBSTITUTION METHOD

Integrals of the form $= \int [f(x)]^n f'(x)\,dx,\ n \neq 1$

1. Evaluate : $\int \sin^3 x \cos x\,dx$

Solution :

$$\int (x^2+x+1)^5 (2x+1)\,dx = \int u^5\,du$$

$$= \frac{u^6}{6} + c$$

$$= \frac{(x^2+x+1)^6}{6} + c$$

Put $u = \cancel{x^2+x+1}\ \sin x$

$$\frac{du}{dx} = \cancel{2x+1}\ \cos x$$

$$du = \cancel{(2x+1)dx}\ \cos x\,dx$$

2. Evaluate : $\int (x^2+x+1)^5(2x+1)\,dx$

Solution :-

$$\int (x^2+x+1)^5(2x+1)\,dx = \int u^5\,du$$

$$= \frac{u^6}{6}+C$$

$$= \frac{(x^2+x+1)^6}{6}+C$$

Put $u = x^2+x+1$

$$\frac{du}{dx} = 2x+1$$

$$du = (2x+1)\,dx$$

3.3 STANDARD INTEGRALS

1. Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2-a^2}$, $\int \frac{dx}{\sqrt{a^2 \pm x^2}}$, $\int \frac{dx}{\sqrt{x^2-a^2}}$

To Evaluate : $\int \frac{dx}{x^2-a^2}$

Consider, $\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a}$

$$\frac{1}{(x+a)(x-a)} = \frac{A(x-a)+B(x+a)}{(x+a)(x-a)}$$

$$\therefore 1 = A(x-a)+B(x+a)$$

Put $x=a$, then $1 = A(0) + B(2a)$

$$\therefore B = \frac{1}{2a}$$

Put $x=-a$, then $1 = A(-2a) + B(0)$

$$\therefore A = -\frac{1}{2a}$$

$$\int \frac{1}{x^2-a^2}\,dx = \int \left[\frac{A}{x+a} + \frac{B}{x-a}\right] dx$$

$$= \int \left[\frac{-\frac{1}{2}a}{x+a} + \frac{\frac{1}{2}a}{x-a}\right] dx$$

$$= \frac{1}{2a} \int \left[\frac{-1}{x+a} + \frac{1}{x-a} \right] dx$$

$$= \frac{1}{2a} \left[-\log (x+a) + \log (x-a) + c \right.$$

$$= \frac{1}{2a} \left[\log \left(\frac{x-a}{x+a} \right) \right] + c$$

$$\therefore \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \log \left[\frac{x-a}{x+a} \right] + c$$

---

1. Evaluate : $\int \frac{dx}{16 + x^2}$

$$\int \frac{dx}{16 + x^2} = \int \frac{dx}{4^2 + x^2}$$

$$= \frac{1}{4} \tan^{-1} \left[\frac{x}{4} \right] + c$$

---

2. Evaluate : $\int \frac{dx}{\sqrt{9 - x^2}}$

$$\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{dx}{\sqrt{3^2 - x^2}} = \sin^{-1} \left[\frac{x}{3} \right] + c$$

UNIT-IV

INTEGRAL CALCULUS-II

4.1 INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du$$

1. Evaluate : $\int x e^x dx$

Solution :-

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$dv = e^x dx$

$\int dv = \int e^x dx$

$v = e^x$

$\int u\,dv = uv - \int v\,du$

$\int x e^x dx = xe^x - \int e^x dx$

$= xe^x - ex + c$

2. Evaluate :- $\int x e^{2x} dx$

Solution :-

Let $u = x$

$du = dx$

$dv = e^{2x} dx$

$v = \int e^{2x} dx = \frac{e^{2x}}{2}$

Integration by parts now gives us

$\cancel{\int x e^{2x}}$ $\int x\, e^{2x} dx = uv - \int v\,du$

$\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x} dx$

$\frac{xe^{2x}}{2} - \frac{1}{2}\left[\frac{e^{2x}}{2}\right] + c = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$

4.2 BERNOULLI'S FORMULA

Bernoulli's formula :-

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

1. Evaluate : $\int x^2 \sin 3x\,dx$

Solution:-

Let $I = \int x^2 \sin 3x\,dx$

Put $u = x^2 \quad \therefore u' = 2x, \quad u'' = 2, \quad u''' = 0$

$dv = \sin 3x\,dx \qquad \therefore v = \frac{-\cos 3x}{3}$

$v_1 = \frac{-\sin 3x}{9}, \quad v_2 = \frac{\cos 3x}{27}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$$

$$\int x^2 \sin 3x\,dx = x^2\left[\frac{-\cos 3x}{3}\right] - 2x\left[\frac{-\sin 3x}{27}\right] + c$$

$$= \frac{-x^2\cos 3x}{3} + \frac{2x\sin 3x}{9} + \frac{2\cos 3x}{27} + c$$

2. Evaluate : $\int x^2 \sin nx\,dx$

Solution:-

Let $I = \int x^2 \sin nx\,dx$

Put $u = x^2 \quad \therefore u' = 2x, \quad u'' = 2, \quad u''' = 0$

$dv = \sin 3x\,dx \qquad \therefore v = \frac{-\cos nx}{n}$

$v_1 = \frac{-\sin nx}{9n^2} \qquad v_2 = \frac{\cos nx}{n^3}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$$

$$\int x^2 \sin nx\,dx = x^2\left[\frac{-\cos nx}{n}\right] - 2x\left[\frac{-\sin nx}{n^2}\right] + 2\left[\frac{\cos nx}{n^3}\right] + c$$

$$= \frac{-x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2\cos nx}{n^3} + c$$

Evaluate : $\int x^2 \sin^2 x\,dx$

Let $I = \int x^2 \sin^2 x\,dx$

$$\int x^2\left[\frac{1-\cos 2x}{2}\right]dx$$

$$= \frac{1}{2}\int (x^2 - x^2\cos 2x)\,dx$$

$$= \frac{1}{2}\left[\int x^2 dx - \int x^2 \cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \int x^2 \cos 2x\,dx\right] \rightarrow ①$$

consider $\int x^2 \cos 2x\,dx$

Put $u = x^2$, $\therefore u' = 2x$, $u'' = 2$

Put $dv = \cos 2x\,dx$, $\therefore v = \frac{\sin 2x}{2}$, $v_1 = \frac{-\cos 2x}{4}$, $v_2 = -\frac{\sin 2x}{8}$

$$\therefore \int u\,dv = uv - u'v_1 + u''v_2 - \dots$$

$$\int x^2 \cos 2x\,dx = x^2\left[\frac{\sin 2x}{2}\right] - 2x\left[\frac{-\cos 2x}{4}\right] + 2\left[\frac{-\sin 2x}{8}\right] + c$$

$$= \frac{x^2 \sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4} + c \rightarrow ②$$

Using ② in ① we get

$$\therefore I = \frac{1}{2}\left[\frac{x^3}{3} - \left[\frac{x^2\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4}\right]\right] + c$$

$$= \frac{1}{2}\left[-\frac{x^3}{3} - \frac{x^2 \sin 2x}{2} - \frac{x \cos 2x}{2} + \frac{\sin 2x}{4}\right] + c$$

4.3.1 DEFINITE INTEGRALS

Definite Integral:-

$$F(b) - F(a)$$

$$\therefore \int_a^b F(x)\,dx = F(b) - F(a)$$

Properties of Definite integrals:-

1. Evaluate : $\int_1^2 \frac{1}{x}\,dx$

Solution:-

$$\int_1^2 \frac{1}{x}\,dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2 - 0$$

$$= \log 2$$

2. Evaluate : $\int_1^2 x^2\,dx$

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{1}{3}\left[2^3 - 1^3\right]$$

$$= \frac{1}{3}(8-1) = \frac{1}{3}(7) = \frac{7}{3}$$

3. Evaluate : $\int_{-1}^{1} (2x^2 - x^3)\,dx$

Solution:-

$$\int_{-1}^{1} (2x^2 - x^3)\,dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \left[\frac{2(1)^3}{3} - \frac{(1)^4}{4}\right] - \left[\frac{2(-1)^3}{3} - \frac{(-1)^4}{4}\right]$$

$$=\left[\frac{2}{3}-\frac{1}{4}\right]-\left[-\frac{2}{3}-\frac{1}{4}\right]$$

$$=\frac{2}{3}-\frac{1}{4}+\frac{2}{3}+\frac{1}{4}=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}.$$

4. Evaluate :- $\int_0^{\pi/2}\frac{\cos^2x}{1+\sin x}\,dx$

$$\int_0^{\pi/2}\frac{\cos^2x}{1+\sin x}\,dx=\int_0^{\pi/2}\frac{(1-\sin^2x)}{1+\sin x}\,dx$$

$$=\int_0^{\pi/2}\frac{(1-\sin x)(1+\sin x)}{(1+\sin x)}\,dx$$

$$=\int_0^{\pi/2}(1-\sin x)\,dx$$

$$=\left(x+\cos x\right)_0^{\pi/2}$$

$$=\left[\left(\frac{\pi}{2}+\cos\frac{\pi}{2}\right]-(0+\cos 0)\right]$$

$$=\left[\left(\frac{\pi}{2}+0\right]-(1)\right]=\frac{\pi}{2}-1$$

5. Evaluate :- $\int_0^1 x\,(1-x)^n\,dx$

Solution :

Let $I\int_0^1 x\,(1-x)^n\,dx$

$$=\int_0^1(1-x)\left[1-(1-x)\right]^n dx \qquad \left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$=\int_0^1(1-x)x^n\,dx$$

$$=\int_0^1(x^n-x^{n+1})\,dx$$

$$= \int_0^1 (x^n - x^{n+1})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{(n+2)-(n+1)}{(n+1)\,(n+2)} = \frac{n+2-n-1}{(n+1)(n+2)}$$

$$\int_0^1 x\,(1-x)^n\,dx = \frac{1}{(n+1)(n+2)}$$

---

6. Evaluate : $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$.

Let $$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\frac{\cos x}{\sin x}}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left[\frac{\pi}{3}+\frac{\pi}{6}-x\right]}}{\sqrt{\sin\left[\frac{\pi}{3}+\frac{\pi}{6}-x\right]}+\sqrt{\cos\left[\frac{\pi}{3}+\frac{\pi}{6}-x\right]}}\,dx$$

$$\left[\text{Since} \int_a^b f(x)\,dx = \int_a^b F(a+b-x)\,dx\right]$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left[\frac{\pi}{2}-x\right]}}{\sqrt{\sin\left[\frac{\pi}{2}-x\right]}+\sqrt{\cos\left[\frac{\pi}{2}-x\right]}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \quad \rightarrow (2)$$

$$(1) + (2) \Rightarrow \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$2I = \int_{\pi/6}^{\pi/3} dx = \left[x\right]_{\pi/6}^{\pi/3}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$I = \frac{1}{2}\left[\frac{\pi}{6}\right] = \frac{\pi}{12}$$

$$\int_{\pi/3}^{x/6} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{12}.$$

5.2 First order differential equation

Introduction:- Differential equation is one of the branch of Mathematics and social sciences.

Differential equation An equation involving one dependant variable and its derivative with respect to one (or) more Independent variables $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots\right)$ is called differential equation.

Ex:- i) $(1+x^2)\frac{dy}{dx^2} + 2x = x^2$

1. **Order of differential equation**

The order of differential equation is the order of the highest order derivative occuring in the diff. eqn

Ex $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1+x^2}} = 0$ order is 1 $\left(\frac{dy}{dx}\right)$

2. **Degree of a diff. eqn** The degree of the differential equation is the degree of highest order derivative of the equation after the diff. eqn is made free from radicals and fractions so for the derivatives are concerned.

Ex $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$, the degree is one since the highest order is 1. (ie) $\frac{dy}{dx}$

1. **solve!** $(1-e^x)\sec^2 y\, dy + 3e^x \tan y\, dx = 0$

Solution Given $(1-e^x)\sec^2 y\, dy + 3e^x \tan y\, dx = 0$

Both sides divide by $(1-e^x)\tan y$ we get

$$\frac{\sec^2 y}{\tan y} dy + 3\frac{e^x\, dx}{1-e^x} = 0$$

Unit - V

Application of Integration

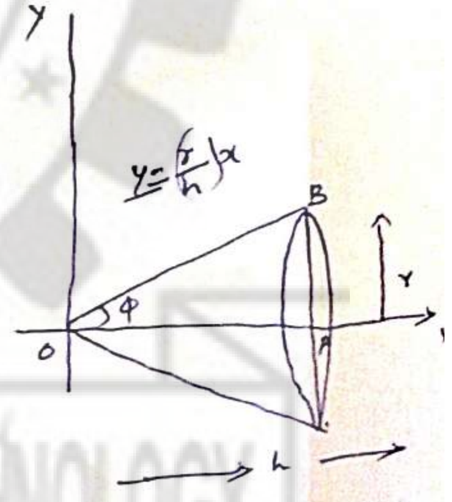
5.1 AREA and VOLUME

Area:- Area bounded by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$

1. Find the volume of a right circular cone of base radius "r" and altitude "h" by Integration.

Solution:- When the $\triangle OAB$ bounded by the line OB and X-axis is roated about X-axis, the solid cone is obtained equation of OB is $y = mx$

$$\therefore y = \frac{r}{h}x \quad \left[m = \tan\phi = \frac{r}{h} \right]$$



volume of cone: $\int_a^b \pi y^2\,dx$

$$= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx$$

$$= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\left(\frac{x^3}{3}\right)_0^h$$

$$= \frac{\pi r^2}{h^2}\left(\frac{h^3}{3}\right)$$

$$= \frac{1}{3}\pi r^2 h$$

volume of the cone $= \frac{1}{3}\pi r^2 h$ cubic units

5.3 LINEAR TYPE DIFFERENTIAL EQUATION

Solution of Linear differential equation.

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

Where. Integrating factor $e^{\int P dx}$.

Solve:-

(1) $\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$$

$P = \cot x \qquad Q = 4x \operatorname{cosec} x$

$$\frac{dy}{dx} + Py = Q$$

$$I.F = e^{\int P dx} = e^{\int \cot x \, dx}$$

$$= e^{\int \frac{\cos x}{\sin x} dx}$$

$$= e^{\log(\sin x)}$$

$$= \sin x$$

$$I.F \quad y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y \sin x = \int 4x \operatorname{cosec} x \sin x \, dx + C$$

$$y \sin x = 4 \int x \frac{1}{\cancel{\sin x}} \cancel{\sin x} \, dx + C$$

$$y \sin x = 4 \int x \, dx + C$$

$$y \sin x = 4 \frac{x^2}{2}$$

$$\boxed{y \sin x = 2x^2 + C}$$

$$\int \frac{\sec^2 y}{\tan y}\, dy + 3 \int \frac{e^x}{1-e^x}\, dx = 0$$

$$\int \frac{\sec^2 y}{\tan y}\, dy = -3 \int \frac{e^x}{1-e^x}\, dx$$

$$\int \frac{\sec^2 y}{\tan y}\, dy = 3 \int \frac{-e^x}{1-e^x}\, dx$$

$$\log \tan y = 3 \log (1-e^x) + \log c$$

$$\log \tan y = \log (1-e^x)^3 + \log c$$

$$\log \tan y = \log \left[c\,(1-e^x)^3\right]$$

$$\tan y = c\,(1-e^x)^3$$

This is required solution.

(2) Solve: $\tan x \sec^2 y\, dy + \tan y \sec^2 x\, dx = 0$

Soln

$$\tan x \sec^2 y\, dy = -\tan y \sec^2 x\, dx$$

$$\int \frac{\sec^2 y}{\tan y}\, dy = -\int \frac{\sec^2 x}{\tan x}\, dx$$

$$\log (\tan y) = -\log (\tan x) + \log c$$

$$\log \tan y + \log \tan x = \log c$$

$$\log (\tan x \tan y) = \log c$$

$$\tan x \tan y = c$$

(5) Solve:-

$$\frac{dy}{dx} + y \tan x = e^x \cos x$$

$$\frac{dy}{dx} + Py = Q \qquad P = \tan x \qquad Q = e^x \cos x$$

$$I.F = e^{\int P dx} = e^{\int \tan x \, dx}$$

$$= e^{\log(\sec x)}$$

$$I.F = \sec x$$

$$y e^{\int P dx} = \int e^x \cos x \sec x \, dx + c$$

$$= \int e^x \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}} \, dx$$

$$y \sec x = \int e^x \, dx$$

$$y \sec x = e^x + c \; //$$