Introduction to materials

Introduction to materials

Without Materials there is No Engineering

Types of Materials

- Materials can be divided into the following categories
  - Crystalline
  - Amorphous

Crystalline Materials

- These are materials containing one or many crystals. In each crystal, atoms or ions show a long range periodic arrangement.
- All metals and alloys are crystalline materials.
- These include iron, steel, copper, brass, bronze, aluminum, duralumin , uranium, thorium etc.

Amorphous Material

- The teim amorphous refers to materials that do not have regular, periodic arrangement of atoms
- Glass is an amorphous material

Another Classification of Materials

Another useful classification of materials is _

- Metals
- Ceramics
- Polymers
- Composites

Major Classes of Materials

- Metals
  - Ferrous (Iron and Steel)
  - Non-ferrous metals and alloys

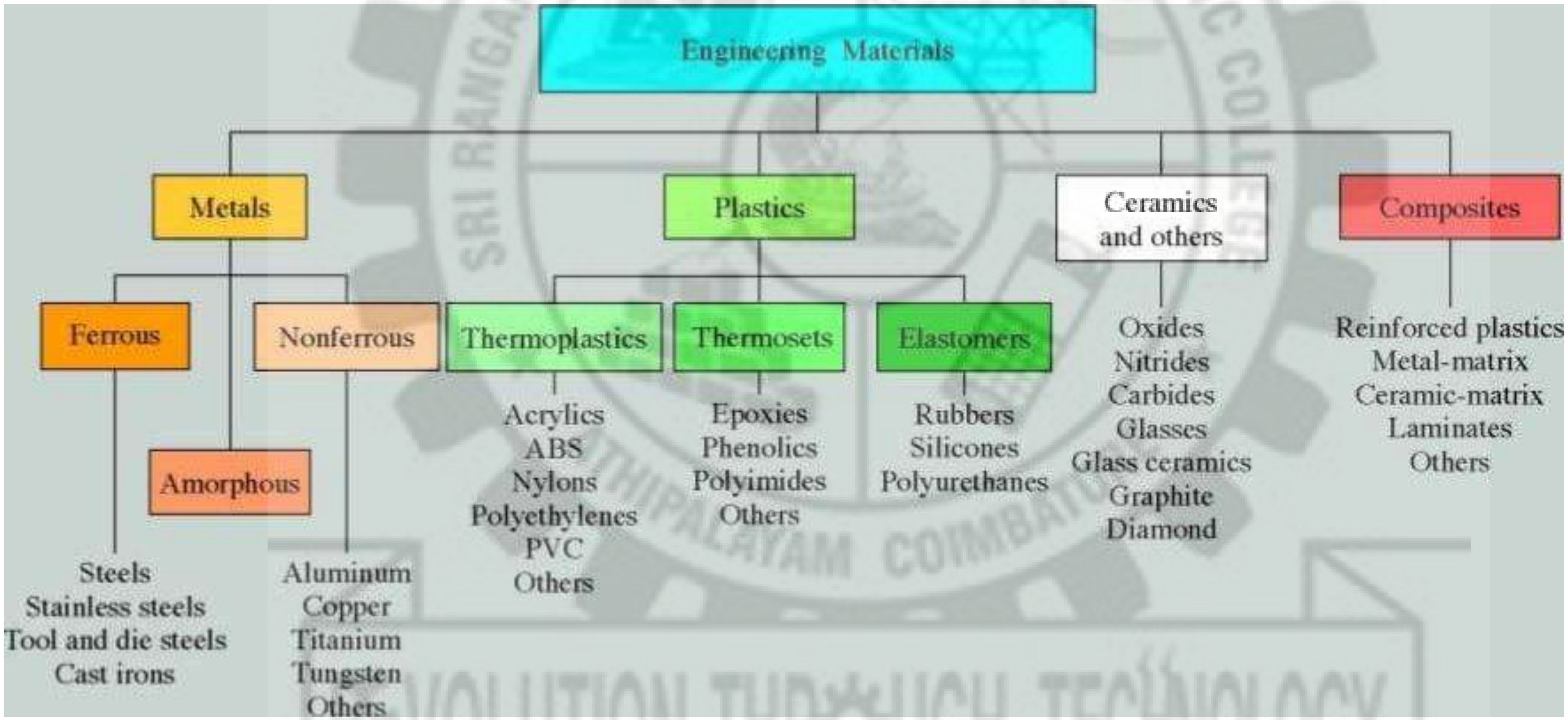
- Ceramics
  - Structural Ceramics (high-temperature toad bearing)
  - Refractories (corrosion-resistant, insulating)
  - Whitewares (e.g. porcelains)
  - Glass
  - Electrical Ceramics (capacitors, insulators, transducers, etc.)
  - Chemically Bonded Ceramics (e.g. cement and concrete)

Six Major Classes of Materials

- Polymers
  - Plastics
  - Elastomers

- Composites
  - Particulate composites (small oarticles embedded in a different material)
  - Laminate composites (golf ctub shafts, tennis rackets, Damaskus swords)
  - Fiber reinforced composites (e.g. fibergtass)

Engineering Materials



Properties of Materials

› An alternative to major classes, you may *divide* materials into classification according to important properties.

› One goal of materials engineering is to select materials with suitable properties for a given application, so it's a sensible approach.

Just as for classes of materials, there is some overlap among the properties, so the divisions are not always clearly defined

- Mechanical properties
- Electrical properties
- Dielectric properties
- Magnetic properties
- Optical properties
- Corrosion properties
- Biological properties

Properties of Materials

Mechanical properties

A. Elasticity and stiPness (recoverable stress vs. strain)
B. Ductility (non-recoverable stress vs. strain)
C. Strength
D. Hardness
E. BrittTeness
F. Toughness
E. Fatigue
F. Creep

Properties of Materials

Electrical properties

A. Electrical conductivity and resistivity

Dielectric properties

A. Polarizability
B. Capacitance
C. Ferroelectric properties
D. Piezoelectric properties
E. Pyroelectric properties

Magnetic properties

A. Paramagnetic properties
B. Diamagnetic properties
C. Ferromagnetic properties

Properties of Materials

Optical properties

A. Refractive index
B. Absorption, reflection, and transmission
C. Birefringence (double refraction)

Co<osion properties

Biological

properties

A. Toxicity
B. bio-compatibility

Mechanical propeties

- Elasticity and stiffness (recoverable stress vs. strain)
- Ductility (non-recoverable stress vs. strain)
- Strength
- Hardness
- Brittleness
- Toughness
- Fatigue
- Creep

Elasticity and stillness

- Elastic deformation is the deformation produced in a material which is fully recovered when the stress causing it is removed.

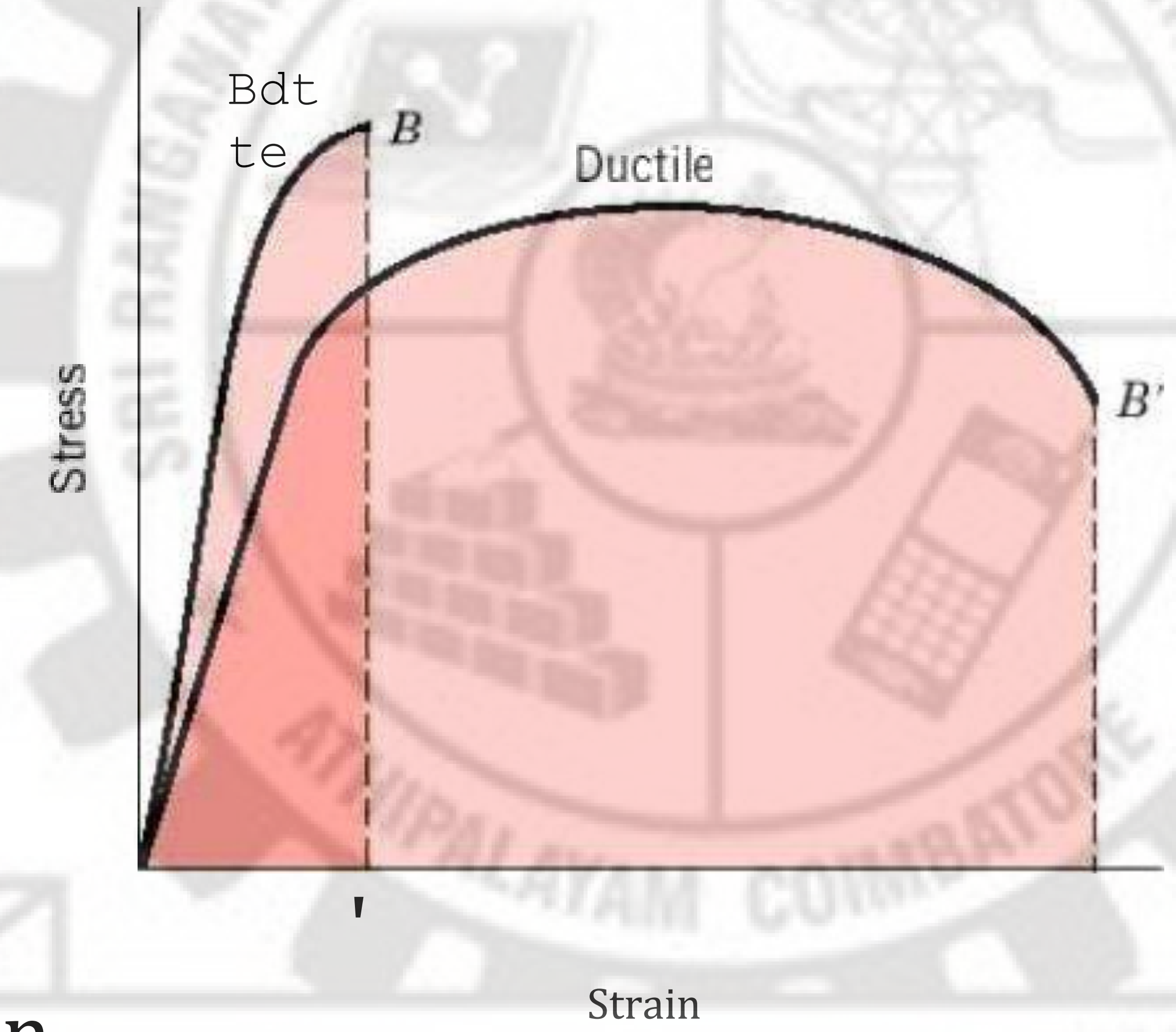
- Stiffness is a qualitative measure of the elastic deformation produced in a material. A stiP material has a high modulus of elasticity.

- Modulus of elasticity or Young’s modulus is the slop of the stress — strain curve during elastic deformation.

Ductility

- Ductility is the ability of the material to stretch or bend permanently without breaking.

Ductility



Ductility is a measure of the deformation at fracture -

Defined by percent elongation or percent reduction
" in area

Strength

- Yield strength is the stress that has to be exceeded so that the material begins to deform plastically.
- Tensile strength is the maximum stress which a material can withstand without breaking.

Hardness

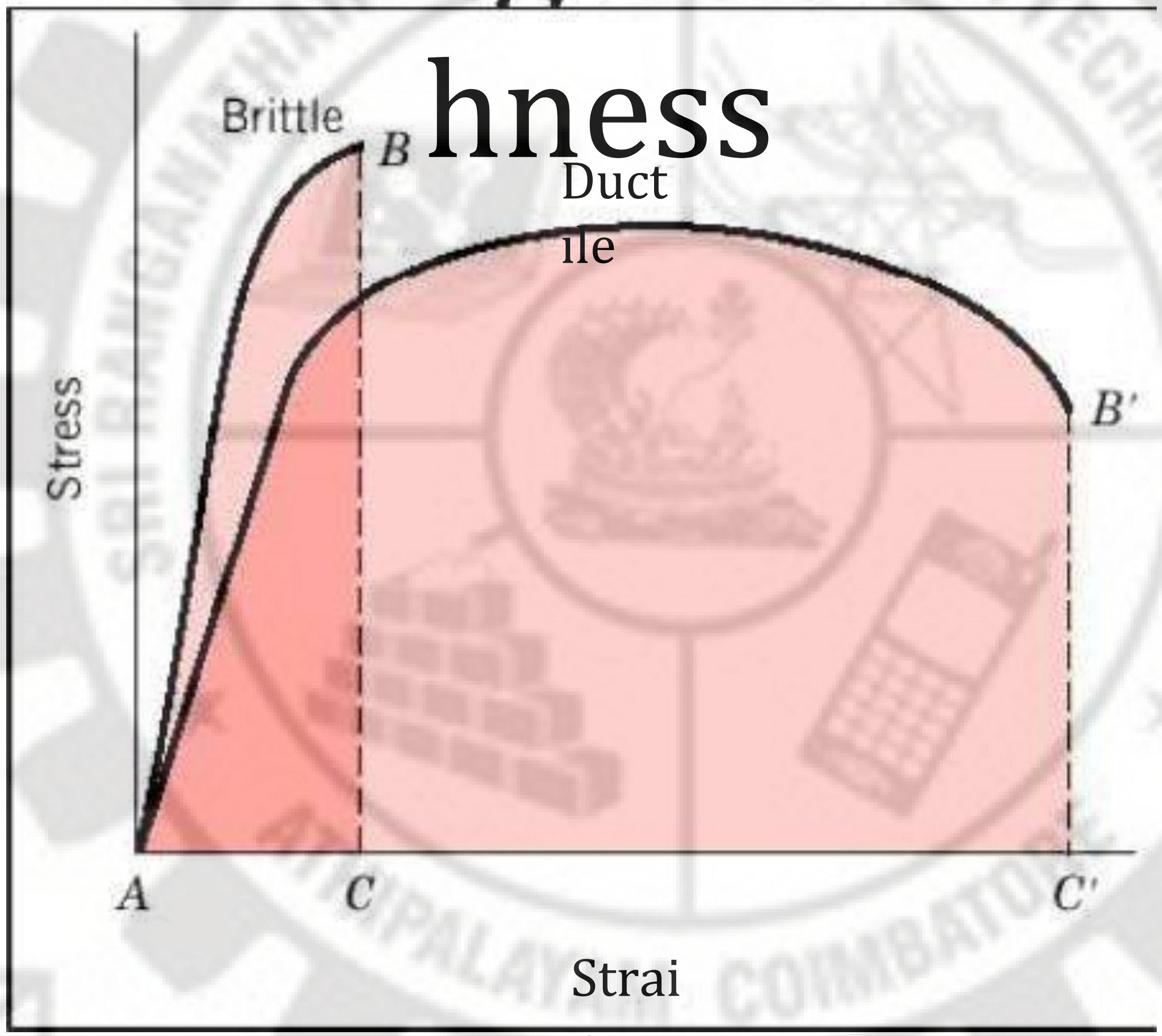
- Hardness is the resistance to penetration of the surface of a material.

Brittleness and Toughness

- The material is said to be brittle if it fails without any plastic deformation

- Toughness is defined as the energy absorbed before fracture.

Toug hness



Toughness = the ability to absorb energy up to fracture

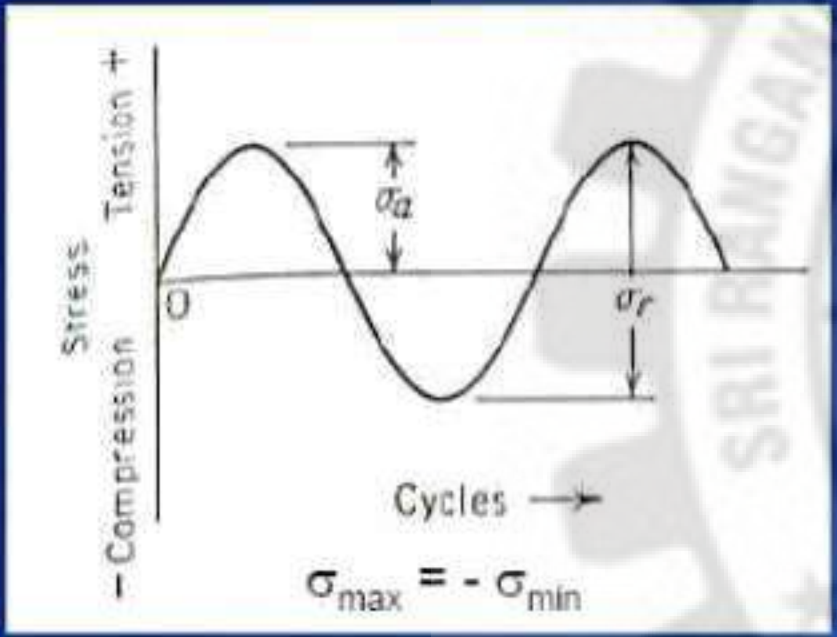
= the total area under the strain-stress curve up to fracture

Fatig
ue

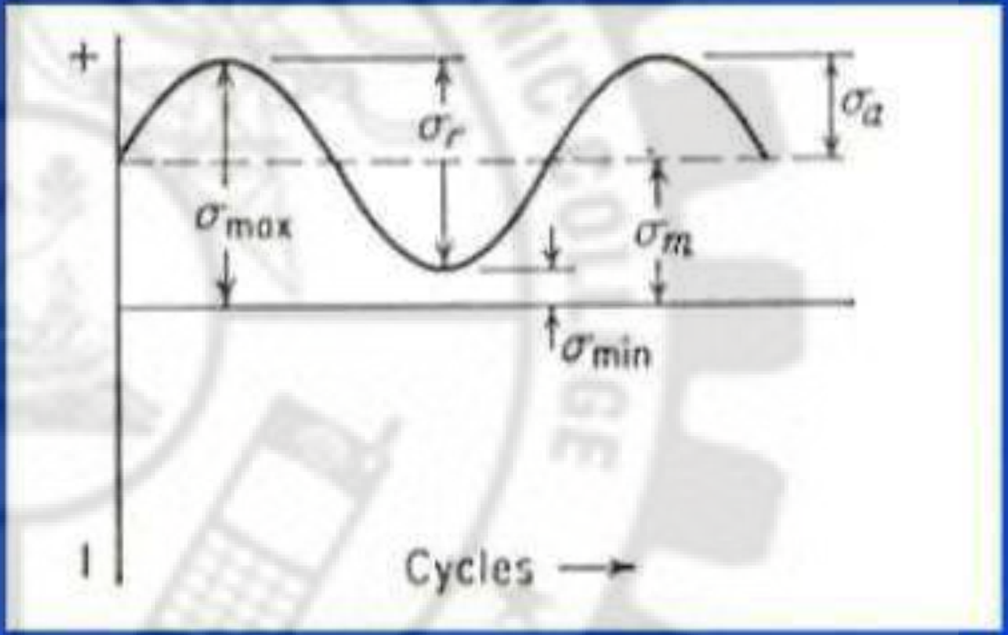
- Fatigue failure is the failure of material under fluctuating load.

Stress cycles

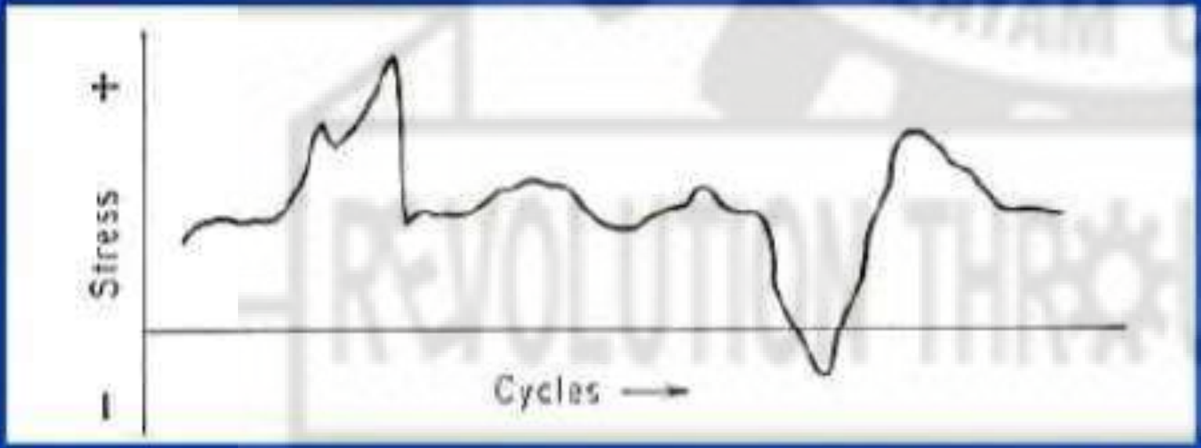
Different types of fluctuating stress



(a) Completely reversed cycle of stress (sinusoidal)



(b) Repeated stress cycle



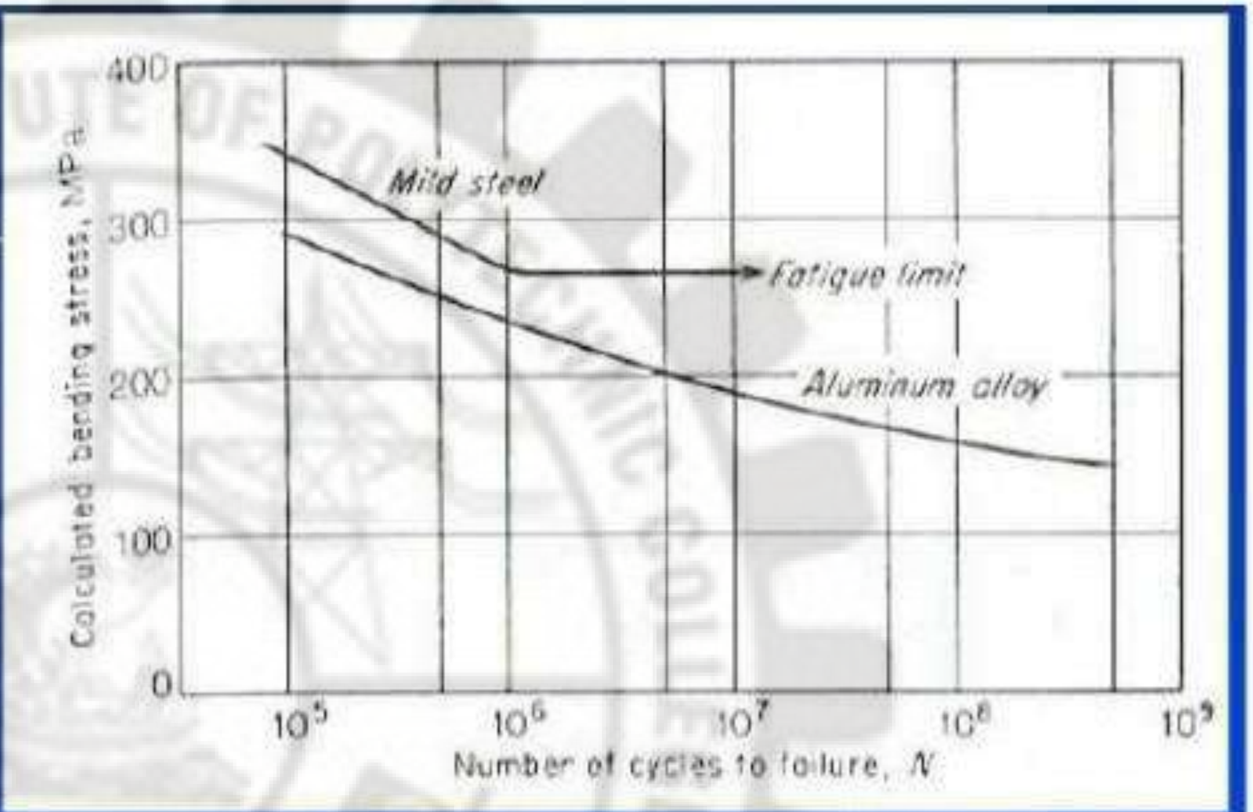
(c) Irregular or random stress cycle

Tensile stress +

Compressive stress -

The S-N curve

- *Engineering fatigue data* is normally represented by means of *S-N curve*, a plot of *stress S* against the *number of cycle, N*.
- *Stress can be* → σ_a, σ_{max}, σ_{min}
- σ_m *, R or A should be mentioned.*



Typical fatigue curves

- *S-N curve* is concerned chiefly with *fatigue failure at high numbers of cycles* (N > 10^5 cycles) → high cycle fatigue (*HCF*).
- *N* < 10^4 or 10^5 cycles → low cycle fatigue (*LCF*).
- *N* increases with decreasing *stress level*.
  - *Fatigue limit or endurance limit* is normally defined at 10^7 or 10^8 cycles. Below this limit, the material presumably can endure an infinite number of cycle before failure.
  - *Nonferrous metal*, i.e., aluminium, do not have *fatigue limit* → fatigue strength is defined at ~ 10^8 cycles.

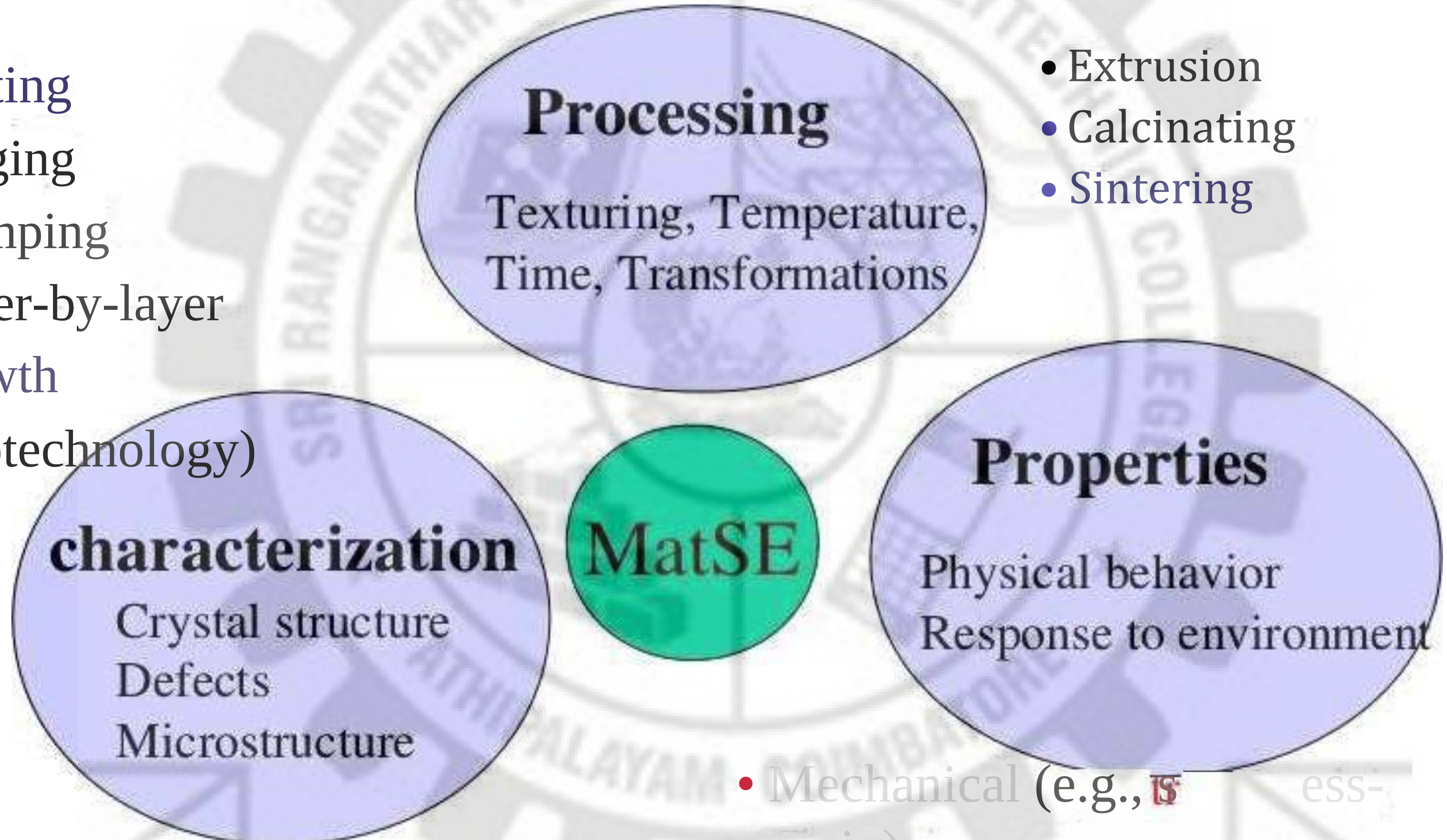
Cre p
e

- Creep is the time dependent peimanent deformation under a constant load at high temperature.

What is Materials Science & Engineering?

- Casting
- Forging
- Stamping
- Layer-by-layer growth (nanotechnology)

- Extrusion
- Calcinating
- Sintering



- Microscopy: Optical, transmission electron, scanning tunneling
- X-ray, neutmn, e- diffraction
- Spectroscopy

- Mechanical (e.g., [illegible] ess- snain)
- Electrical
- Magnetic
- Optical
- Cmr‹nive
- Deterimative characteristics

Metals

Metals

- Metals can be classified as
  - Ferrous
    - Ferrous material include iron and its alloys (steels and cast irons)
  - Non-ferrous
    - Non-ferrous materials include all other metals and alloys except iron and its alloys.
    - Non-ferrous materials include Cu, Al. Ni etc. and their alloys such as brass, bronze, duralumin etc.

Ferrous metals and alloys

- Steel
  - Steels are alloys of iron and carbon in which carbon content is less than 2%. Other alloying elements may be present in steels.
- Cast iron
  - Cast irons are alloys of iron and carbon in which carbon content is more than 2%. Other alloying elements may be present in cast irons.

Stee
l

- Steels are alloys of iron and carbon in which <u>carbon content is less than 29o. </u>Other alloying elements may be present in steels.
- They may be classified as
  - Plain carbon steel
  - Alloy steel

Plain Carbon Steel

These are alloys of iron containing only carbon up to 2%. Other alloying elements may be present in plain carbon steels as impurities.

They can be further classified as

1. Low carbon steel (< 0.3% C)
2. Medium carbon steel (0.3 – 0.5% C)
3. High carbon steel (> 0.5% C)

Alloy Steel

These are alloys of iron containing carbon up to 2% along with other alloying elements such as Cr, Mo, W etc. for specific properties.

They can be further divided on the basis of total alloy content fOther than carbonJ present in them as given below.

- —Low alloy steel (Total alloy content < 2H)
- —Medium alloy steel (Total alloy content 2 - 59a)
- —High alloy steel (Total alloy content > 59)

Cast iron

- Cast irons are alloys of iron and carbon containing more than 2% carbon. They may also contain other alloying elements.
- They can be further divided as below
  - White cast iron
  - Grey cast iron
  - Malleable cast iron
  - S.G. iron

Cast iron

- White cast iron contains carbon in the form of cementite (Fe C).
- Grey cast iron contains carbon in the form of graphite flakes.
- Malleable cast iron is obtained by heat treating white cast iron and contains rounded clumps of graphite formed from decomposition of cementite.
- S.G. iron contain carbon in the form of spheroidal graphite particles during solidification. It is also known as nodular cast iron.

Non-ferrous Metals and Alloys

- Non-ferrous Metals and Alloys include all other metals and alloys except iron and its alloys.
- Non-ferrous Metals and Alloys include Cu, Al, Ni etc. and their alloys such as
  - — Brass (alloy of Cu-Zn)
  - — Bronze (alloy of Cu —Sn)
  - — Duralumin (alloy of Al-Cu) etc.

Classes and Properties: Metals

Distinguishing features

- Atoms arranged in a regular repeating structure (crystalline)
- Relatively good strength
- Dense
- Malleable or ductile: high plasticity
- Resistant to fracture: tough
- Excellent conductors of electricity and heat
- Opaque to visible light
- Shiny appearance

- Thus, metals can be formed and machined easily, and are usually long-lasting materials.

- They do not react easily with other elements,

•One of the main drawbacks is that metals do react with chemicals in the environment, such as iron-oxide (corrosion).

- Many metals do not have high melting points, making them useless for many applications.

Classes and Properties: Metals

Applications

- Electrical wiring
- Structures: buildings, bridges, etc.
- Automobiles: body, cnassis, springs, engine btock, etc.
- Airplanes: engine components, fuselage, landing gear assembly, etc.
- Trains: raits, engine comDonents, body, wheels
- Machine tools: drill bits, hammers, screwdrivers, saw blades, etc.
- Magnets
- Catalysts

Examples

- Pure metal elements (Cu, Fe, Zn, Ag, etc.)
- Alloys (Cu-Sn=bronze, Cu-Zn=brass, Fe-C=steet, Pb-Sn=sotder,)

Ceramics

Types of Ceramic s

- Structural Ceramics (high-temperature toad bearing)
- Refractory (corrosion-resistant, insulating)
- White wares (e.g. porcelains)
- Glass
- Electrical Ceramics {capacitors, insulators, transducers, etc.)
- Chemically Bonded Ceramics (e.g. cement and concrete)

Classes and Properties: Ceramics

Distinguishing features

- Except for glasses, atoms are regulaily arranged (crystalline)
- Composed of a mixture of metal and nonmetal atoms
- Lower density than most metals
- Stronger than metals
- Low resistance to fracture: low toughness or brittle
- Low ductility or malleability: low plasticity
- High melting point
- Poor conductors of electricity and heat
- Single crystals are transparent

- Where metals react readily with chemicals in the environment and have low application temperatures in many cases, ceramics do not suffer from these drawbacks.

- Ceramics have high-resistance to environment as they are essentially metals that have already reacted with the environment, e.g. Alumina (AI_2O,) and Silica (SiO_2, Quartz).

- Ceramics are heat resistant. Ceramics form both in crystalline and non-crystalline phases because they can be cooled rapildy from the molten state to form glassy materials.

Classes and Properties: Ceramics

Applications

- Electrical insulators
- Abrasives
- Thermal insulation and coatings
- Windows, television screens, optical fibers (glass)
- Corrosion resistant applications
- Electrical devices: capacitors, varistors, transducers, etc.
- Highways and roads (concrete)
- Biocompatible coatings (fusion to bone)
- Self-lubricating bearings
- Magnetic materials (audio/video tapes, hard disks, etc.)
- Optical wave guides
- Night-vision

Examples

- Simple oxides (SiO_2 At,O„ Fe,O_3 MgO)
- Mixed-metal oxides (SrTiO„ $MgAt,O_4$ YBa,Cu,O „ having vacancy defects.)
- Nitrides (Si, N_4 AtN, GaN, BN, and TiN, which are used for hard coatings.)

Polymer
s

Polymer
s

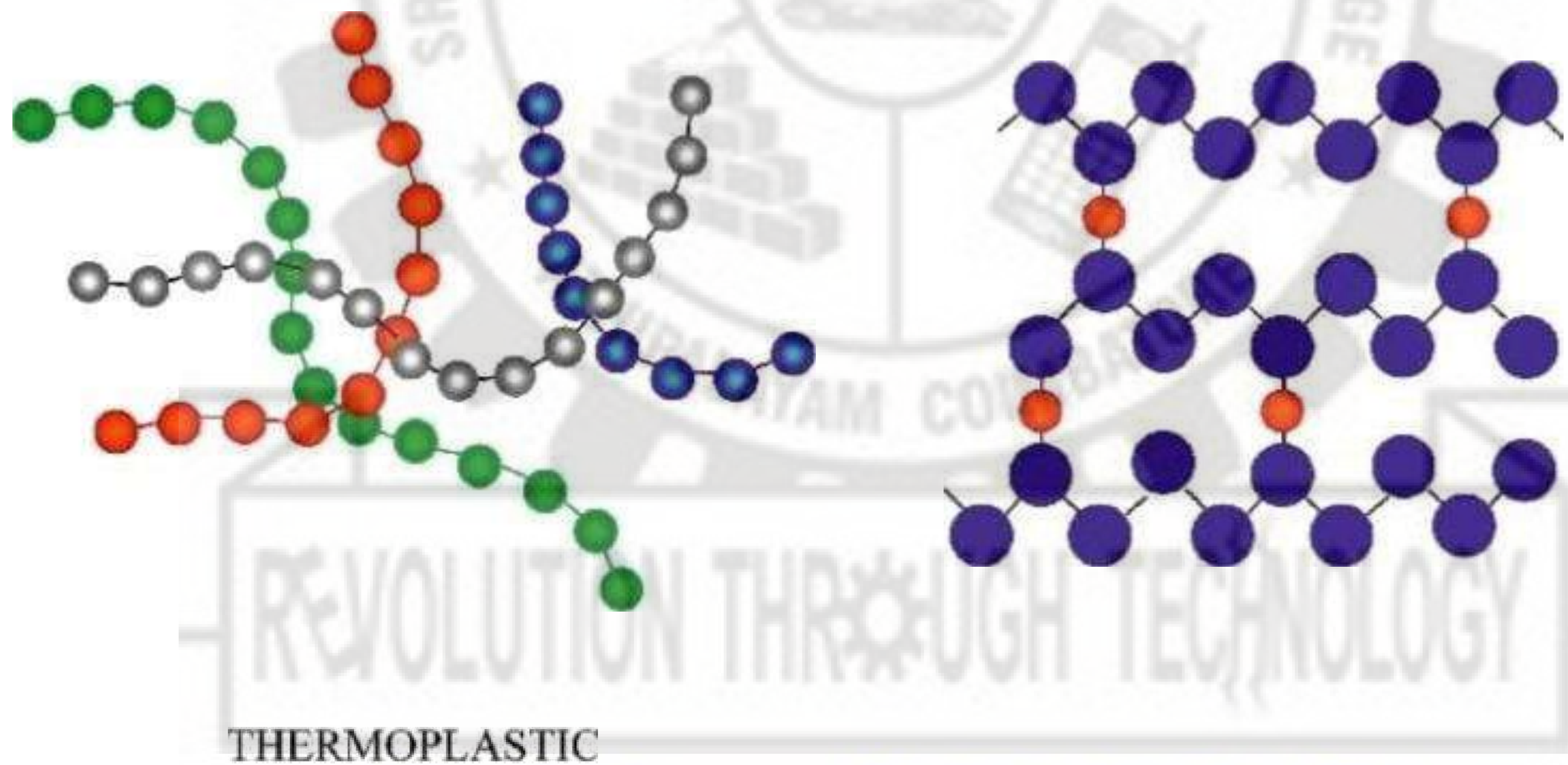
- Plastics
  - Thermoplastics (acrylic, nylon, polyethylene, ABS,. . .
  - Thermosets (epoxies, Polymides, Phenolics, ...

- Elastomers (rubbers, silicones, polyurethanes, ...

Classes and Properties: Polymers

Two main fypes *of polymers* are *thermosets and* the/zrioplastics.

- Thermoplastics are long-chain polymers that slide easily past one another when heated, hence, they tend to be easy to form, bend, and break.
- Thermosets are cross-linked polymers that form 3-D networks, hence are strong and rigid.



Classes and Properties: Polymers

Distinguishing features

- Composed primarily of C and H (hydrocarbons)
- Low melting temperature.
- Some are crystals, many are not.
- Most are poor conductors of electricity and heat.
- Many have high plasticity.
- A few have good elasticity.
- Some are transparent, some are opaque

•Polymers are attractive because they are usually lightweight and inexpensive to make, and usually very easy to process, either in molds, as sheets, or as coatings.

• Most are very resistant to the environment.

• They are poor conductors of heat and electricity, and tend to be easy to bend, which makes them very useful as insulation for electrical wires.

Classes and Properties: Polymers

Applications and Examples

- Adhesives and glues
- Containers
- Moldable products (computer casings, telephone handsets, disposable razors)
- Clothing and upholstery material (vinyls, polyesters, nylon)
- Water-resistant coatings (latex)
- Biodegradable products (com-starcn packing "peanuts')
- Liquid crystals
- Low-friction materials (tef ton)
- Synthetic oils and greases
- Gaskets and 0-rings (rubber)
- Soaps and surfactants

Composite
s

Composites

- A group of materials foimed from mixtures of metals, ceramics and polymers in such a manner that unusual combinations of properties are obtained.
- Examples are
  - Fibreglass
  - Cermets
  - RCC

Composite
s

Types of Composites:

- Polymer matrix composites
- Metal matrix composites,
- Ceramic matrix composites

Classes and Properties: Composites

Distinguishing features

- Composed of two or more different materials (e.g., metal/ceramic, polymer/polymer, etc.)
- Properties depend on amount and distribution of each type of material.
- Collective properties more desirable than possible with any individual material.

Applications and Examples

- Sports equipment (golf club shafts, tennis rackets, bicycle frames)
- Aerospace materials
- Thermal insulation
- Concrete
- "Smart" materials (sensing and responding)
- Brake materials

Examples

- Fiberglass {glass fibers in a polymer)
- Space shuttle heat shields (interwoven ceramic fibers)
- Paints (ceramic particles in latex)
- Tank armor (ceramic particles in metal)

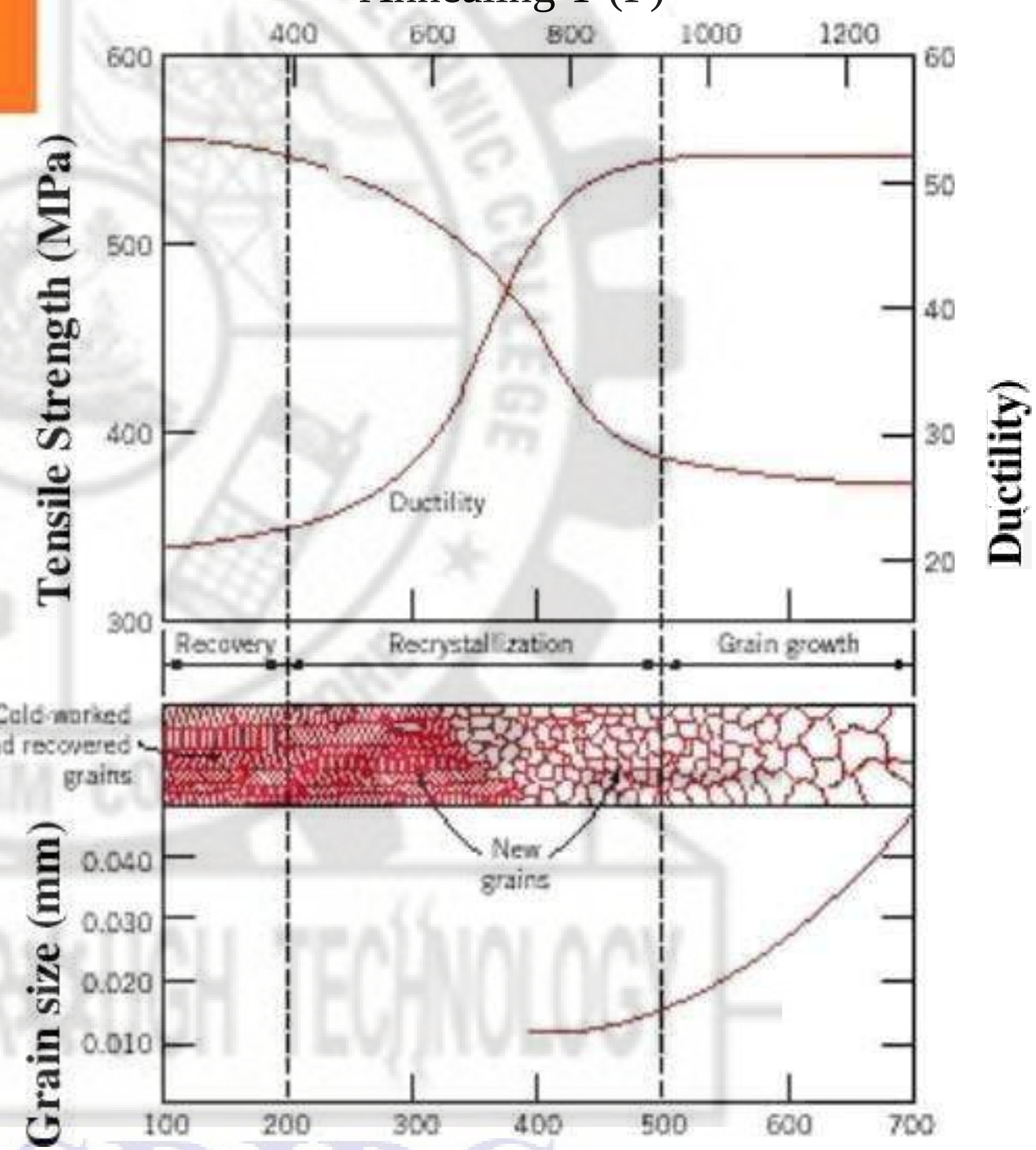
Structure, Properties & Processing

- Properties depend on structure
- Processing for structural changes

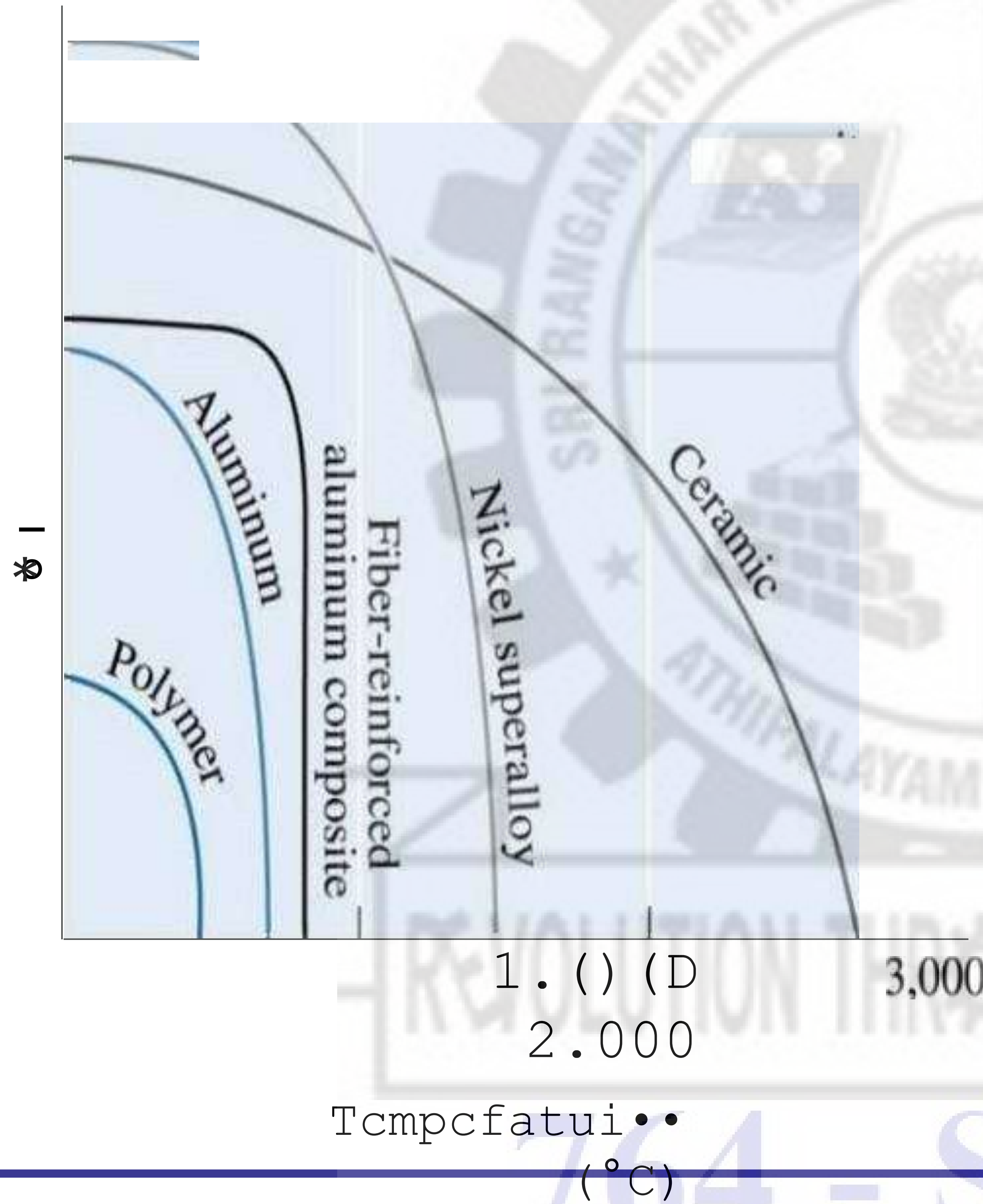
Can you correlate structure and strength and ‹:luctility

Strength versus Structure of Brass and changes in microstructure

Callister: Figs. 21 c-d and 22



Annealing T (C)



Increasing temperature normally reduces the strength of a material.

Polymers are suitable only at low temperatures.

Some composites, special aPoys, and ceramics, have excellent properties at high temperatures

Figure 1.13 Skin operating temperatures for aircraft have increased with the development of improved materials. (After M. Steinberg, Scientific American, October, 19g6.)

Strength-to-weight ratio

D Density is mass per unit volume of a material, usually expressed in units of g/cm or lb/in.

– Strength-to-weight ratio is the strength of a material divided by its density; materials with a high strength-to-weight ratio are strong but lightweight.

TABLE 1-2 *Strength-to-weight ratios of various materials*

Material	Strength ($lb/in.^2$)	Density ($lb/in.^3$)	Strength-to-weight ratio (in.)
Polyethylene	1,000	[illegible]	[illegible]
Pure aluminum	6,500	[illegible]	[illegible]
Al_2O_3	30,000	[illegible]	[illegible]
Epoxy	[illegible]	[illegible]	[illegible]
Heat-treated alloy steel	[illegible]	[illegible]	[illegible]
Heat-treated aluminum alloy	[illegible]	[illegible]	[illegible]
Carbon-carbon composite	[illegible]	[illegible]	[illegible]
Heat-treated titanium alloy	[illegible]	[illegible]	[illegible]
Kevlar-epoxy composite	65,000	[illegible]	[illegible]
Carbon-epoxy composite	80,000	[illegible]	1.60×10^6

Electrical: Resistivity of Copper

Factors affecting electrical resistance

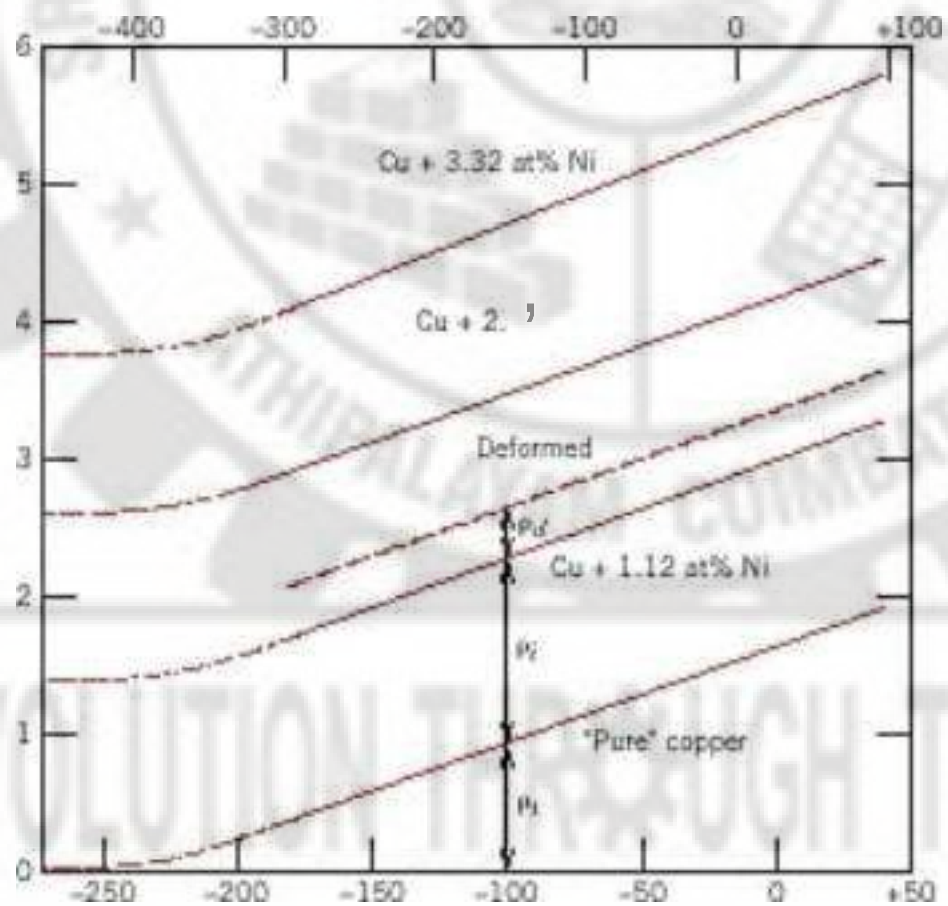
- Composition
- Mechanical deformation
- Temperature

Electrical: Resistivity of Copper

Effect of temperature

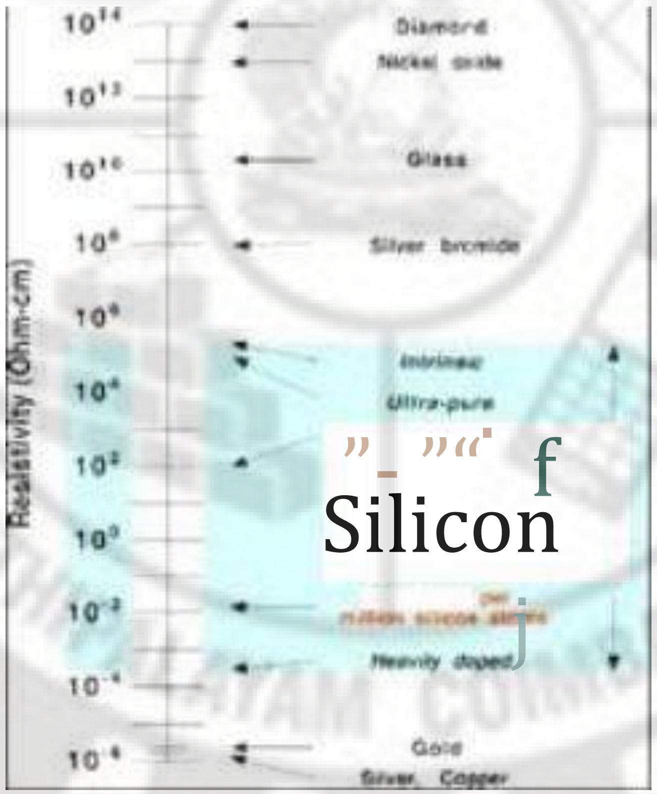
Resistivity

10* Ohms-m



'T(^C)

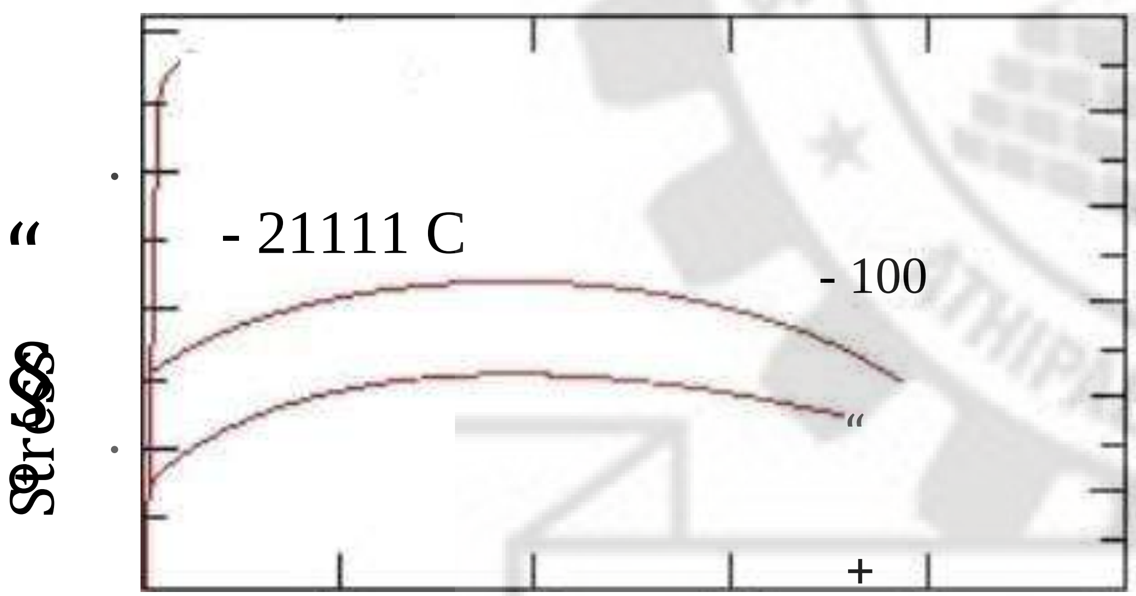
Electrical Conductivity



Deterioration and Failure

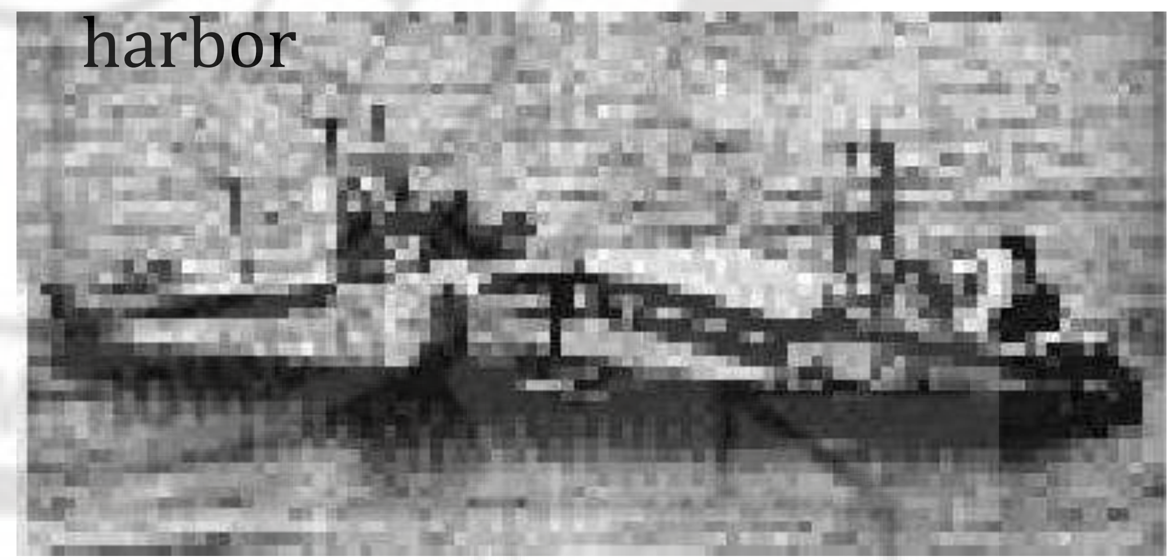
e.g., Stress, corrosive environments, embrittlement, incorrect structures from improper alloying or heat treatments,

bcc Fe Fig. 6.14 Callister



Strain 25 C

USS Esso Manhattan 3/29/43 Fractured at entrance to NY harbor



http://www.uh.edu/iiberty/photos/liberty summary.html

Goals

- Understand the *origin* and *relationship* between processing, structure, properties, and performance.'
- Use the right material for the right job .
- Help recognize within your discipline the design opportunities oPered by materials selection.

While nano -, bfo -, *smart- materials* can make technological revolution, *conservation and re-use methods and policies* can have tremendous environmental and technological impacts!

Motivation: Materials and Failure

Without the right material, a good engineering design is wasted. Need the right material for the right job!

- Materials properties then are responsible for helping achieve engineering advances.
- Failures advance understanding and material's design.
- Some examples to introduce topics we will learn.

The COMET: first jet passenger plane - 1954

- In 1949, the COMET aircraft was a newly designed, modern jet aircraft for passenger travel. It had bright cabins due to large, square windows at most seats. It was composed of light-weight aluminum.
- In early 19 0's, the planes began falling out of the sky.

These tragedies changed the way aide *were designed and the materials*

that were used.

- The square windows were *a "stress comcenfrafor"* and the aluminum alloys used were not *"strong'* enough to withstand the stresses.
- Until them *material selection for mechanical design* was not really considered in designs.

- A Concorde aircraft, one of the most reliable aircraft of our time, was taking oP from Paris Airport when it burst into flames and crashed killing all on board.

- Amazingly, the pilot knowingly steered the plane toward a less populated point to avoid increased loss of life. Only three people on the ground were killed.

- Investigations determined that a jet that took-off ahead of Concorde had a *fatigue-induced* loss of a metallic component of the aircraft, which was left on runway. During take-off, the Concorde struck the component and catapulted it into the wing containing filled fuel tanks. From video, the tragedy was caused from the spewing fuel catching fire from nearby engine exhaust flames and damaging flight control.

Alloying and Diffusion: Advances and Failures

- *Alloying* can lead to new or enhanced properties, e.g. Li, Zr added to Al (advanced *precipitation hardened* 767 aircrañ skin).

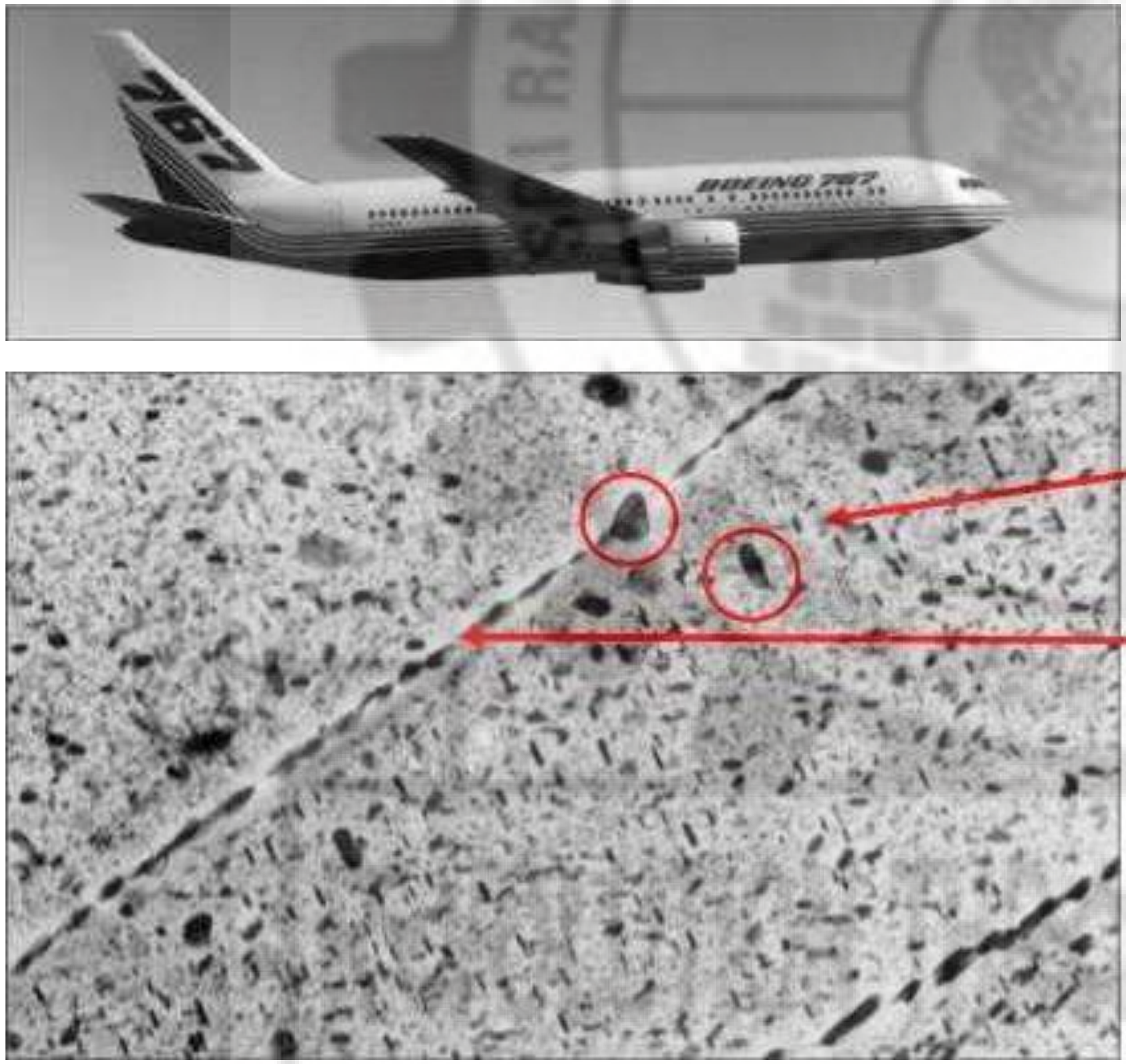
- It can also be a problem, e.g. *Ga is a ‹'asf diHuser at Al grain boundaries* and make Al catastrophically *brittle* (no *plastic behavior* vs. *strain).*

- Need to know *T vs. composition phase diagrams* for what alloying does.

- Need to know *T-T-T (temp - time - transformation) diagrams* to know treatment.

T.J. Anderson and I. Ansara, J. Ph ase Equilibria, 12(1), 64-72 (1991)

Alloying and Precipitation: T-vs- c and TTT diagram



Impacting mechanical response through:

- Precipitates from alloying Al with Li, Zr, Hf,...
- Grain Boundaries

W y, fern Pallister and Rethwisch, Ed. 3
Chapter 11

Conclusions

- **Engineering Requires Consideration of Materials**

  The right materials for the job - sometimes need a new one.

- We will learn about the fundamentals of

  Processing ➔ Structure ⎝ Properties ⎝ Performance

- We will learn that **sometime only simple considerations of property requirements chooses materials.**

  Consider in your engineering discipline what materials that

Unit – II

Chapter 4. DEFORMATION OF METALS

1. Introduction

No engineering material is perfectly rigid. When a material is subjected to external load, it undergoes deformation. While undergoing deformation, the particles of the material exert a resisting force. When this resisting force equals applied load, the equilibrium condition exists hence deformation stops. This internal resistance is called the *stress*.

1. Behaviour of material when subjected to load.

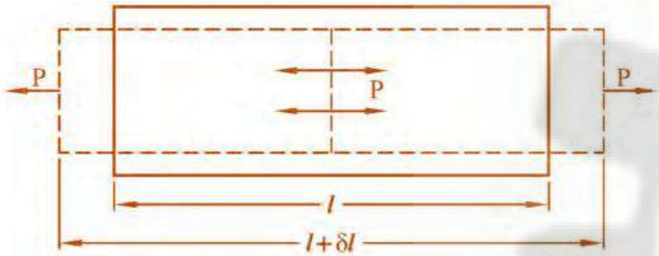


Fig.4.1 Behaviour of material when subjected to load

Consider a bar of uniform cross sectional area A and length l subjected to an axial pull of P at the ends as shown in the fig.4.1.

Consider a section X–X normal to the longitudinal axis of the bar. Due to the action of axial pull, the length of the bar is increased from l to l + 6l and lateral dimension will decrease. In order to keep this section in equilibrium, internal resistance are set up in the section. To avoid separation of the bar at this section, the internal resistance must be equal to the applied load. This internal resistance offered by the section against the deformation is called *stress*.

4.3 Definition of load, stress and strain

Load

The system of external forces acting on a body or structure is known as *load*.

Stress

The stress or intensity of stress at a section may be defined as *the ratio of the internal resistance or load acting on the section to the cross sectional area of that section.*

$$\text{Stress, } f = \frac{\text{Internal resistance}}{\text{Area of cross section}} = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

The unit of stress is N/mm^2. The latest S.I unit for stress is **Pascal.**

1 Pascal	=	1 Pa =	1 N/m^2		=	1×10^{-6} N/mm^2
1 Kilo Pascal	=	1 KPa	= 1×10^3 N/m^2		=	1×10^{-3} N/mm^2
1 Mega Pascal	=	1 MPa	=	1×10^6 N/m^2	=	1 N/mm^2
1 Giga Pascal	=	1 GPa	=	1×10^9 N/m^2	=	1×10^3 N/mm^2

Strain

Strain may be defined as *the ratio between the deformation produced in a body due to the applied load and the original dimension*.

$$\textbf{Strain, } e = \frac{\text{Change in dimension}}{\text{Original dimension}}$$

The strain is only the ratio between the two same quantities and hence it has no unit.

4. Classification of force system

According to the *applied load*, the force system is classified as follows:

1) Tensile stress 2) Compressive stress 3) Shear stress

4) Bending stress 5) Torsional stress

1) Tens

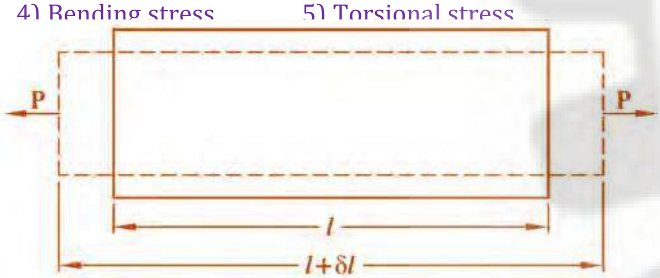


Fig.4.2 Tensile stress

When a load is such that it tends to pull apart the particles of the material causing increase in length in the direction of application of load, then the load is called *tensile load*. The resistance offered against this increased length is called *tensile stress*. The corresponding strain is called *tensile strain*.

$$\textbf{Tensile stress, } f = \frac{\text{Axial pull}}{\text{Area of cross section}} = \frac{P}{A} \ (N/mm^2)$$

$$\textbf{Tensile strain, } e = \frac{\text{Increase in length}}{\text{Original length}} = \frac{\delta l}{l}$$

2) Compressive stress

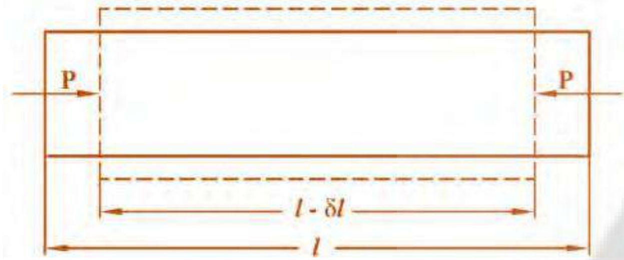


Fig.4.3 Compressive stress

When a load is such that it pushes the particles of the material nearer causing decrease in length in the direction of application of load, then the load is called *compressive load*. The resistance offered against this decrease in length is called *compressive stress* and the corresponding strain is called *compressive strain*.

$$\text{Compressive stress, } f = \frac{\text{Axial push}}{\text{Area of cross section}} = \frac{P}{A} \ (N/mm^2)$$

$$\text{Compressive strain, } e = \frac{\text{Decrease in length}}{\text{Original length}} = \frac{\delta l}{l}$$

3) Shear stress

When a body is subjected to two equal and opposite forces acting tangentially across the resisting section, the body tends to be sheared off across the cross section. Such forces are called *shear force*. The stress induced in the section due to the shear force is called *shear stress* and the corresponding strain is called *shear strain*.

$$\text{Shear stress, } q = \frac{\text{Total shear force}}{\text{Area of resisting section}} = \frac{P}{A} \ (N/mm^2)$$

$$\text{Shear strain, } e = \frac{\text{Change in dimension}}{\text{Original dimension}}$$

4) Bending stress

When a beam is loaded with some external forces, bending moments and shear forces are set up. The bending moment at a section tends to bend or deflect the beam. Internal stresses are developed to resist the bending. These stresses are called *bending stresses.*

5) Torsional stresses

When a machine member is subjected with two equal and opposite couples acting in parallel planes, then the member is said to be in torsion. The stress induced by this torsion is called *torsional stress*.

4.5 Hooke's law

Hooke's law states that *stress is directly proportional to strain within elastic limit.*

i.e. **stress ∝ strain** (or) $\frac{\textbf{Stress}}{\textbf{Strain}}$ **= A constant**

For tensile and compressive stresses, the constant is known as *Young's modulus* or *modulus of elasticity*.

For shear stress, the constant is known as *modulus of rigidity*.

6. Young's modulus or modulus of elasticity

The ratio of stress to strain in tension or compression is known as *Young's modulus* or *modulus of elasticity.* It may also be defined as the slope of stress – strain curve in elastic region. It is denoted by '*E*' and the unit is N/mm^2.

Young's modulus is the measure of stiffness of the material. A member made of material with larger value of Young's modulus is said to have higher stiffness. The stiffer materials undergo smaller deformation for a given load condition.

6. Working stress

The maximum stress to which the material of a member or machine element is subjected in normal usage is called *working stress.* It is also known as *allowable stress* or *design stress*. To avoid permanent set, the working stress is kept less than the elastic limit.

6. Factor of safety and load factor

The ratio of ultimate stress to working stress is known as *factor of safety*.

$$\textbf{Factor of safety} = \frac{\textbf{Ultimate stress}}{\textbf{Working stress}}$$

The value of factor of safety varies from 3 in case of steel to as high as 20 in case of timber subjected to suddenly applied load. The value of factor safety depends on the following factors.

1) The reliability of the material
2) The accuracy with which the maximum load on the member is determined
3) The nature of loading
4) The effect of corrosion and wear
5) The effect of temperature
6) Possible manufacturing defects.

Load factor: The ratio of ultimate load to working load is called load factor.

$$\text{Load factor} = \frac{\text{Ultimate load}}{\text{Working load}}$$

4.9 Linear strain or longitudinal strain

Linear strain or longitudinal strain is defined as the ratio of the change in length to the original length.

$$\text{Linear or longitudinal strain, } e = \frac{\text{Change in length}}{\text{Original length}} = \frac{6l}{l}$$

4.10 Deformation due to tensile or compressive force

Consider a bar subjected to an axial pull or push at the ends. Due to this load, deformation occurs in the bar.

Let, P = Load acting on the bar

l = Length of the bar

A = Cross sectional area of the bar

ƒ = Stress induced in the bar

e = Strain in the bar

6l = Deformation of the bar and

E = Young's modulus of the material of the bar

According to Hooke's law,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{-------------} (1)$$

$$\text{Stress, } f = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

$$\text{Strain, } e = \frac{\text{Change in length}}{\text{Original length}} = \frac{6l}{l}$$

Substituting the values of stress and strain in equation (1)

$$\frac{\left(\frac{P}{A}\right)}{\left(\frac{6l}{l}\right)} = E$$

$$\boxed{6l = \frac{P\,l}{A\,E}} \quad \text{(or)} \quad \boxed{6l = \frac{f\,l}{E}} \quad \left(\because \frac{P}{A} = f\right)$$

4.11 Bars of varying sections

Consider a bar having different cross sections for different length as shown in the fig.4.5. Let this bar is subjected to an axial pull or push at the

ends. It may be noted that each section in the bar is subjected to the same axial push or pull. Due to this variations in cross sectional area, the stresses, strain and hence change in length for each section are differe... are cal... nges in length... gths of the sec...

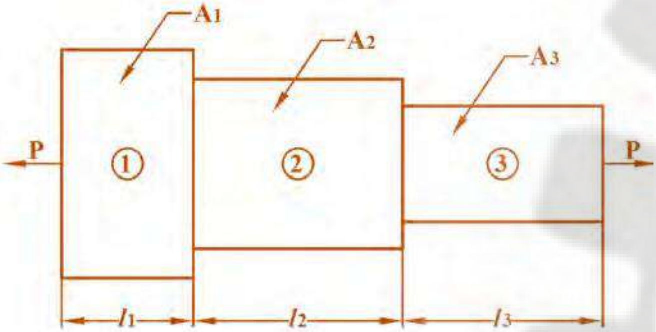


Fig.4.4 Bars of varying sections

Let l_1, l_2, l_3 and A_1, A_2, A_3 be the length and area of the sections of 1, 2, 3 respectively.

Change in length of section 1, $\delta l_1 = \frac{P l_1}{A_1 E}$

Similarly, $\delta l_2 = \frac{P l_2}{A_2 E}$; $\delta l_3 = \frac{P l_3}{A_3 E}$

Total deformation of the bar, $\delta l = \delta l_1 + \delta l_2 + \delta l_3$

$$= \frac{P l_1}{A_1 E} + \frac{P l_2}{A_2 E} + \frac{P l_3}{A_3 E} = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right)$$

If the modulus of elasticity is different for different sections, then

$$\delta l = P\left(\frac{l_1}{A_1 E_1} + \frac{l_2}{A_2 E_2} + \frac{l_3}{A_3 E_3}\right)$$

4.12 Shear stress and shear strain

When a body is subjected to two equal and opposite forces acting tangentially across the resisting section, the body tends to be sheared off

across the cross section. Such forces are called *shear force*. The stress induced in the section due to the shear force is called *shear stress* and the corresponding strain is called *shear strain*. In shear, the strain is measured by the angle in radians through which the body is distorted by the applied force.

Consider a cube ABCD of side l fixed at the bottom face DC. Let a tangential force P be applied at the face AB. As a result of this force, the cube is ... angle ϕ as shown in fig

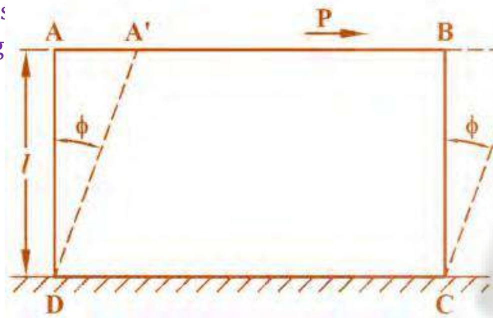


Fig.4.5 A body subjected to shear force

$$\text{Shear strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{DA' - DA}{DA} = \frac{AA'}{l} = \sin\phi = \phi$$

(∵ For small angle, $\sin\phi = \phi$)

4.13. Modulus of rigidity or shear modulus

The ratio of shear stress to shear strain within the elastic limit is known a *modulus of rigidity* or *shear modulus*. It is denoted by N or G or C and the unit is N/mm^2. Larger is the modulus of rigidity, lesser is the distortion when a body is subjected to shear stress.

$$\textbf{Modulus of rigidity, } G = \frac{\textbf{Shear stress}}{\textbf{Shear strain}}$$

14. Lateral strain

It is the ratio of *the change in lateral dimension to the original dimension.* Lateral strain is induced along the direction perpendicular to the direction of application of load.

14. Poisson's ratio

The ratio of the lateral strain to the corresponding longitudinal strain within elastic limit is called Poisson's ratio. It is represented by n (nu) or 1/m.

$$\textbf{Poisson's ratio} = \frac{1}{m} = \frac{\textbf{Lateral strain}}{\textbf{Longitudinal strain}}$$

For most of the material, Poisson's ratio lies between 0.25 to 0.33.

4.16 Volumetric strain

When a body is subjected to an axial pull or push, it undergoes change in its dimensions and hence its volume will also change.

The ratio of change in volume to the original volume is known as

volumetric strain

Volumetric strain, e_v **=** $\frac{\textbf{Change in volume}}{\textbf{Original volume}}$ = $\frac{6Y}{Y}$

4.17 Bulk modulus

When a body is subjected to three mutually perpendicular stresses of same magnitude, the ratio of the direct stress to the corresponding volumetric strain is known as *bulk modulus* or *bulk modulus of elasticity*. It represents the resistance of a body against volumetric strain. It is usually denoted by K.

Bulk modulus, K **=** $\frac{\textbf{Direct stress}}{\textbf{Volumetric strain}}$ = $\frac{p}{e_v}$

4.18 Volumetric strain of various sections

1) Rectangular bar

Fig.4.6 Volumetric strain in rectangular bar

Consider a rectangular bar of length l, width b and thickness t and is subjected to an axial tensile force P as shown in fig.4.7.

Let 6l, 6b, 6t be the changes in dimensions due to the applied load.

Original volume, Y_1 = (b × t × l)

Final volume, Y_2 = (b + 6b)(t + 6t)(l + 6l)

= (b + 6b)(tl + t6l + l6t + 6l 6t)

= (b t l + b t 6l + b l 6t + b 6l 6t + t l 6b + t 6l 6b + l 6t 6b + 6b 6l 6t)

Neglecting the higher powers of δl, δb and δt,

Final volume, Y_2 = b t l + b t 6l + b l 6t + t l 6b

Change in volume, 6Y = Final volume – Original volume

= b t l + b t 6l + b l 6t + t l 6b – b t l

$= b\ t\ ðl + b\ l\ ðt + t\ l\ ðb$

Volumetric strain = $\frac{\text{Change in volume}}{\text{Original volume}}$

$$\frac{6Y}{Y} = \frac{b\,t\,6l + b\,l\,6t + t\,l\,6b}{b\,t\,l} = \frac{6l}{l} + \frac{6t}{t} + \frac{6b}{b}$$

But, $\frac{6l}{l}$ = Longitudinal strain = e

$\frac{ðt}{t}$ = Lateral strain = $-\frac{1}{m}$ e (∵ Thickness decreases)

$\frac{ðb}{b}$ = Lateral strain = $-\frac{1}{m}$ e (∵ Width decreases)

Volumetric strain = $e - \frac{1}{m}e - \frac{1}{m}e = e - \frac{2e}{m}$

$$\frac{ðV}{V} = e\left(1 - \frac{2}{m}\right)$$

Change in volume, $6Y = e\left(1 - \frac{2}{m}\right)Y$

2) Circular bar

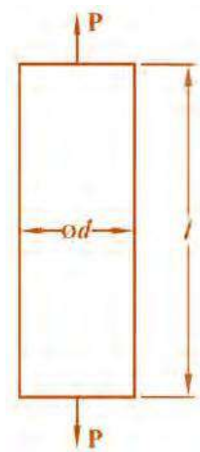


Fig.4.7 Volumetric strain in circular bar

Consider a circular bar of diameter d and length l and is subjected to a tensile force of P as shown in fig.4.8.

Let 6d and 6l be the change in dimension due to the applied load.

Original volume, $Y_1 = \frac{v}{4} d^2 l$

Final volume, $Y_2 = \frac{v}{4}[(d + 6d)^2 \times (l + 6l)]$

$$= \frac{v}{4}[(d^2 + 2d6d + 6d^2) \times (l + 6l)]$$

$$= \frac{v}{4} \; ^2 \; (d\,l + d\,6l + 2\,d\,l\,6d + 2\,d\,6l\,6d + l\,6d^2 + 6d^2\,6l)$$

Neglecting the higher powers of δd and δ*l*

$$Y_2 = \frac{v}{4}(d^2l + d^26l + 2\,d\,l\,6d)$$

Change in volume, δV = Final volume – Original volume

$$= \frac{v}{4}(d^2l + d^26l + 2\,d\,l\,6d) - \frac{v}{4} d^2l$$

$$= \frac{v}{4}(d^26l + 2\,d\,l\,6d)$$

Volumetric strain, $e_v = \frac{6Y}{Y} = \frac{\text{Change in volume}}{\text{Original volume}}$

$$= \frac{\frac{v}{4}(d^26l + 2\,d\,l\,6d)}{\frac{v}{4}d^2l} = \frac{d^26l}{d^2l} + \frac{2\,d\,l\,6d}{d^2l}$$

$$= \frac{6l}{l} + \frac{2\,6d}{d}$$

But, $\frac{6l}{l}$ = Longitudinal strain = e

$\frac{\delta d}{d}$ = Lateral strain = $-\frac{1}{m}$ e (∵ Diameter decreases)

Volumetric strain, $\frac{\delta V}{V} = e + 2\left(-\frac{1}{m}e\right) = e\left(1 - \frac{2}{m}\right)$

Change in volume, $6Y = e\left(1 - \frac{2}{m}\right)Y$

4.19 Relation between Young's modulus (E) and modulus of rigidity (N)

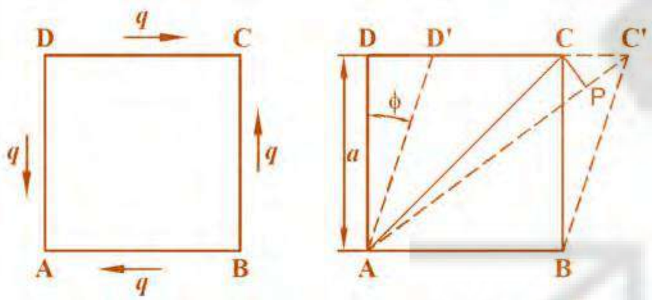


Fig.4.8 Relation between E and C

Consider a square element ABCD of side 'a' and unit thickness. Let the element is distorted to ABC'D' due to shear stress 'q' acting as shown in the fig.4.9. Due to the shear stress, the diagonal AC will be elongated and the diagonal BD will be shortened.

Linear strain of diagonal AC, $= \frac{q}{E} - \frac{1}{m}\left(-\frac{q}{E}\right)$

Linear strain of diagonal AC, $= \frac{q}{E}\left(1 + \frac{1}{m}\right)$ – – – – – (1)

Let this shear stress q cause shear strain ϕ resulting in the diagonal AC to distort to AC'.

Strain along diagonal AC $= \frac{\text{Change in length}}{\text{Original length}}$

$$= \frac{AC' - AC}{AC} = \frac{AC' - AP}{AC} \quad (\because AC = AP)$$

$$= \frac{PC'}{AC} \quad \text{– – – – – (2)}$$

From triangle CC'P, $PC' = CC' \sin 45^0 = \frac{CC'}{\sqrt{2}}$

$$AC = \sqrt{AD^2 + CD^2} = \sqrt{2\,CD^2} = \sqrt{2}\,CD \quad (\because AD = CD)$$

Substitute the values of PC' and AC in equation (2)

Linear strain of diagonal AC $= \frac{CC'}{\sqrt{2}\,\sqrt{2}\,CD} = \frac{CC'}{2\,CD} = \frac{1}{2}\frac{CC'}{CD}$

From triangle CC'B, $\tan\phi = \frac{CC'}{BC} = \frac{CC'}{CD}$ $(\because BC = CD)$

Since the angel is very small, tan ϕ = ϕ

$$\therefore \phi = \frac{CC'}{CD}$$

$$\frac{q}{C} = \frac{CC'}{CD} \quad \left(\because \text{Shear strain, } \phi = \frac{q}{C}\right)$$

$\therefore$ Linear strain of diagonal AC, $= \frac{1}{2}\frac{q}{C}$ – – – – – (3)

Combining equation (1) and (3)

$$\frac{q}{E}\left(1 + \frac{1}{m}\right) = \frac{1}{2}\frac{q}{C}$$

$$\left(1 + \frac{1}{m}\right) = \frac{E}{2C}$$

$$E = 2C\left(1 + \frac{1}{m}\right)$$

4.20 Relation between bulk modulus (K) and Young's modulus (E)

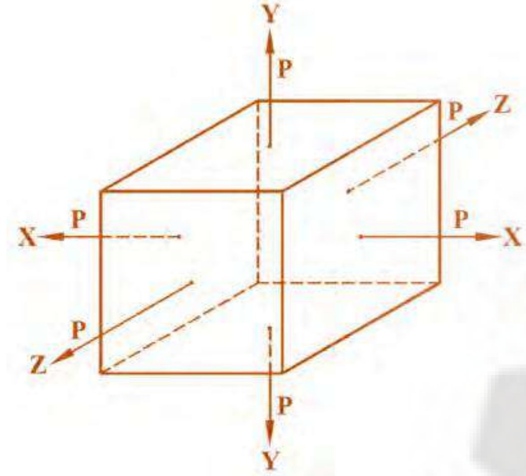


Fig.4.9 Relation between K and E

Consider a cube subjected to three mutually perpendicular tensile stresses of equal intensity as shown in fig.4.10.

Let, f be the stress acting on each face of the cube.

The strain in x direction, $e_x = \frac{f_x}{E} - \frac{f_y}{mE} - \frac{f_z}{mE}$

$$e_x = \frac{f}{E}\left(1 - \frac{2}{m}\right) \quad (\because f_x = f_y = f_z = f)$$

Similarly, $e_y = \frac{f}{E}\left(1 - \frac{2}{m}\right)$ and $e_z = \frac{f}{E}\left(1 - \frac{2}{m}\right)$

Volumetric strain, $\frac{\delta V}{V} = e_x + e_y + e_z = 3\frac{f}{E}\left(1 - \frac{2}{m}\right)$

Bulk modulus, $K = \frac{\text{Direct stress}}{\text{Volumetric strain}}$

$\therefore$ Volumetric strain $= \frac{\text{Direct stress}}{\text{Bulk modulus}} = \frac{f}{K}$

$$3 \times \frac{f}{E}\left(1 - \frac{2}{m}\right) = \frac{f}{K}$$

$$\frac{3}{E}\left[1 - \frac{2}{m}\right] = \frac{1}{K}$$

$$\Rightarrow E = 3K\left(1 - \frac{2}{m}\right)$$

4.21 Relation between E, C and K

We know that, $E = 2C\left(1 + \frac{1}{m}\right)$ ------------(1)

Also, $E = 3K\left(1 - \frac{2}{m}\right)$ ------------(2)

Equating (1) and (2)

$$2C\left(1 + \frac{1}{m}\right) = 3K\left(1 - \frac{2}{m}\right)$$

$$2C + \frac{2C}{m} = 3K - \frac{6K}{m}$$

$$\frac{6K}{m} + \frac{2C}{m} = 3K - 2C$$

$$\frac{1}{m}(6K + 2C) = 3K - 2C$$

$$\frac{1}{m} = \frac{3K - 2C}{6K + 2C}$$

Substituting the value of $\frac{1}{m}$ in equation (1)

$$E = 2C\left(1 + \frac{3K - 2C}{6K + 2C}\right)$$

$$= 2C\left(\frac{6K + 2C + 3K - 2C}{6K + 2C}\right)$$

$$= \frac{2C}{2}\left(\frac{9K}{3K + C}\right)$$

$$\boxed{E = \frac{9KC}{3K + C}}$$

4.22 Composite bars

A composite bar may be defined as *a bar made of two or more different materials joined together in such a way that the system elongates or contracts as a whole equally when subjected to axial pull or push.*

Consider a composite bar made of two different materials as shown in the fig.4.11

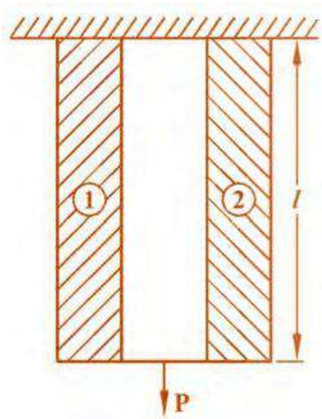


Fig.4.10 Composite bar

Let, P = Total load on the bar

l = Length of the bar

A_1 = Area of bar 1

E_1 = Young's modulus of bar 1

P_1 = Load shared by bar 1 and

A_2, E_2, P_2 are corresponding values for bar 2

According to the definition of composite bar,

THE STRAIN IN BOTH THE MATERIAL IS SAME.

i.e. $\frac{f_1}{E_1} = \frac{f_2}{E_2}$

$$f_1 = \frac{E_1}{E_2} f_2$$

The ratio $\frac{E_1}{E_2}$ is known as *modular ratio*

Total load, P = Load shared by bar 1 + Load shared by bar 2

$$P = P_1 + P_2$$

$$= f_1 A_1 + f_2 A_2$$

$$= \frac{E_1}{E_2} f_2 A_1 + f_2 A_2$$

$$= \frac{E_1 f_2 A_1 + E_2 f_2 A_2}{E_2}$$

$$P = \frac{f_2 (E_1A_1 + E_2A_2)}{E_2}$$

$$f_2 = P \left(\frac{E_2}{E_1A_1 + E_2A_2}\right)$$

$$P_2 = f_2A_2 = P \left(\frac{E_2A_2}{E_1A_1 + E_2A_2}\right)$$

Similarly,

$$P_1 = f_1A_1 = P \left(\frac{E_1A_1}{E_1A_1 + E_2A_2}\right)$$

Note: The following points should be remembered while solving the problems in composite bars

1) ***Extension or contraction of the bar being equal and hence the strain is also equal***
2) ***The total external load applied on the composite bar is equal to the sum of the loads shared by the different materials.***

23. Temperature stresses and strains.

When the temperature of a body is increased, it undergoes deformation leading to increase in dimensions. On the other hand the body
contracts when its temperature is reduced.

When a body is allowed to deform freely under increased or reduced temperature condition, stresses are not induced. If the deformation is prevented completely or partially, stresses will be induced in the body.

The stresses induced in a body due to change in temperature are known as *temperature stress* or *thermal stress*. The corresponding strain in the body is known as *temperature strain* or *thermal strain*.

23. Exp **ain**

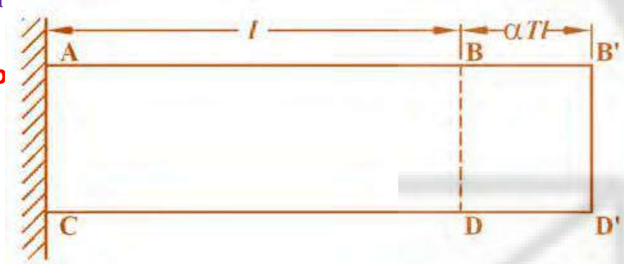


Fig. 4.11 Temperature stress and strain

Consider a body subjected to an increase in temperature. Let, l = Original length of the body

T = Increase in temperature and

a = Co efficient of linear expansion

Increase in length due to increase of temperature, 6l = a Tl

If both the ends of the bar are rigidly fixed so that its expansion is prevented, then compressive stress is induced in the body.

$$\text{Strain, } e = \frac{\text{Change in length}}{\text{Original length}} = \frac{aTl}{l} = aT$$

Stress, ƒ = Strain × Young's modulus = aTE

If the supports yield by an amount equal to λ, then

the actual expansion that has taken place, 6l = aTl – S

$$\text{Strain, } e = \frac{\text{Change in length}}{\text{Original length}} = \frac{aTl - S}{l} = aT - \frac{S}{l}$$

$$\text{Stress, } f = \text{Strain} \times \text{Young's modulus} = \boxed{\left(aT - \frac{S}{l}\right)E}$$

4.25 Strain energy or resilience due to axial load

When a body is subjected to an external load, there is deformation of the body which causes movement of the applied load. Thus work is done by the applied load. This work done is stored in the body as energy and that is why when the load is removed, the body regains its original shape and size behaving like a spring. *This energy stored in the body by virtue of strain is called strain energy or resilience.*

Analytical derivation of strain energy

Consider a body of length l and uniform cross section A and is subjected to an external load P . The deformation takes place from zero to final value of the magnitude, if the load is increased gradually.

Consider an elemental strip of thickness dδ and at a distance δ_1 from the origin. The work done by the external load P for the displacement of d6 is given by,

6w = Load × Displacement = P . d6 ------------- (1)

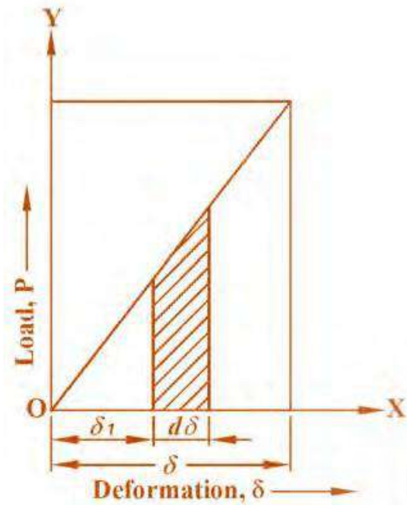


Fig.4.12 Strain energy

We know that, deformation, $6 = \frac{Pl}{A E}$

$$P = \frac{A E 6}{l}$$

Substitute the value of P in equation (1)

$$6w = \frac{A E}{l} 6 . d6$$

Total work done $= \int_0^6 \frac{AE}{l} 6 . d6 = \frac{AE}{l}\left[\frac{6^2}{2}\right]_0^6 = \frac{AE}{l}\left[\frac{6^2}{2}\right]$

Substituting $6 = \frac{fl}{E}$

Total work done $= \frac{AE}{l} \left[\frac{\left(\frac{fl}{E}\right)^2}{2} \right] = \frac{AE}{l} \left[\frac{f^2 l^2}{2E^2}\right] = \frac{f^2}{2E} \times Al = \frac{f^2}{2E} \times \text{Volume}$

But total work done on the bar = Strain energy stored in the bar

∴ The strain energy stored,

$$U = \frac{f^2}{2E} \times \text{Volume}$$

4.26 Proof resilience

The maximum strain energy which can be stored in a body without permanent deformation is called its proof resilience. If p_{max} be the maximum stress at the elastic limit, then

$$\text{Proof resilience} = \frac{f_{max}^2}{2E} \times \text{Volume}$$

4.27 Modulus of resilience

The maximum strain energy which can be stored in a body per unit volume is known as modulus of resilience.

$$\text{Modulus of resilience} = \frac{f_{max}^2}{2E}$$

4.28 Instantaneous stresses due to various types of loads

1. Gradually applied load

Consider a bar subjected to a gradually applied load.

Let, P = Gradually applied load,

A = Cross sectional area of the bar,

l = Length of the bar,

δl = Deformation of the bar

E = Young's modulus of the material of the bar and

f = Instantaneous stress induced in the bar

Since the load is applied gradually, the magnitude of he load is increasing from zero to the final value P .

$$\text{Average load} = \frac{\text{Minimum load} + \text{Maximum load}}{2} = \frac{0 + P}{2} = \frac{P}{2}$$

Work done by the load = Average load × Deflection

$$= \frac{P}{2} \times \delta l$$

The strain energy stored in the bar , U $= \frac{f^2}{2E} \times A\,l$

But strain energy stored = Work done

$$\frac{f^2}{2E} \times A\,l = \frac{P}{2}$$

We know that, $\delta l = \frac{fl}{E}$

$$\therefore \frac{f^2}{2E} \times A\,l = \frac{P}{2} \times \frac{fl}{E}$$

$$f \times A = P$$

$$f = \frac{P}{A}$$

Instantaneous stress produced due to gradually applied load, $f = \frac{P}{A}$

2. Suddenly applied load

Consider a bar subjected to a suddenly applied load.

Let, P = Suddenly applied load,

A = Cross sectional area of the bar,

l = Length of the bar,

δl = Deformation of the bar

E = Young's modulus of the material of the bar and

f = Instantaneous stress induced in the bar

Since the load is applied suddenly, it is constant throughout the process of deformation of the bar.

Work done by the load = Average load × Deflection = P × δl

The strain energy stored in the bar, U $= \frac{f^2}{2E} \times A\,l$

But strain energy stored = Work done

$$\frac{f^2}{2E} \times A\,l = P \times \delta l$$

We know that, δl $\delta l = \frac{fl}{E}$

$$\therefore \frac{f^2}{2E} \times A\,l = P \times \frac{fl}{E}$$

$$\frac{f}{2} \times A = P$$

$$f = 2 \times \frac{P}{A}$$

Instantaneous stress produced due to suddenly applied load, $f = 2 \times \frac{P}{A}$

3. Impact by gravity

Consider a bar in which a collar is attached at the bottom. Let this bar is subjected to a load applied with impact as shown in the fig.4.14.

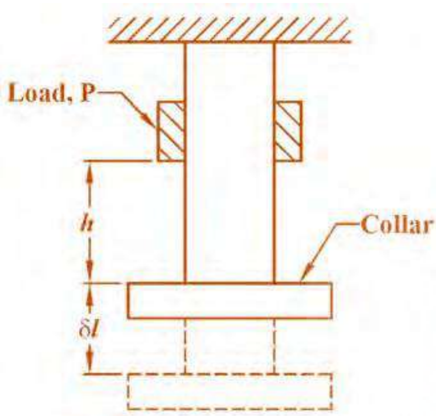


Fig.4.13 Impact by gravity

Let, P = Load applied with impact

A = Cross sectional area of the bar,

l = Length of the bar,

δl = Deformation of the bar due to the load

E = Young's modulus of the material of the bar and

f = Instantaneous stress induced in the bar

h = Height of fall of load before it strikes the collar

Work done by the load = Average load × Distance moved

$$= P(h + \delta l)$$

The strain energy stored in the bar, U $= \frac{f^2}{2E} \times A l$

But strain energy stored = Work done

$$\frac{f^2}{2E} \times A l = P(h + \delta l)$$

We know that, $\delta l = \frac{fl}{E}$

$$\therefore \frac{f^2}{2E} \times A l = P\left(h + \frac{fl}{E}\right)$$

Multiply by $\frac{2E}{Al}$ on both sides,

$$\frac{f^2 \times Al}{2E} \times \frac{2E}{Al} = \frac{2E}{Al} \times \left(Ph + \frac{Pfl}{E}\right)$$

$$f^2 = \frac{2EPh}{Al} + 2f\left(\frac{P}{A}\right)$$

$$f^2 - 2f\left(\frac{P}{A}\right) = \frac{2EPh}{Al}$$

Add $\frac{P^2}{A^2}$ on both sides

$$f^2 - 2f\left(\frac{P}{A}\right) + \frac{P^2}{A^2} = \frac{P^2}{A^2} + \frac{2EPh}{Al}$$

$$\left(f - \frac{P}{A}\right)^2 = \frac{P^2}{A^2} + \frac{2EPh}{Al}$$

Taking square root on both sides, we get,

$$f - \frac{P}{A} = \sqrt{\left(\frac{P^2}{A^2} + \frac{2EPh}{Al}\right)}$$

$$\boxed{f = \frac{P}{A} + \sqrt{\left(\frac{P^2}{A^2} + \frac{2EPh}{Al}\right)}}$$

δl is very small as compared to h,

then Work done = P h

But strain energy stored = Work done

$$\frac{f^2}{2E} \times A\,l = Ph$$

$$f^2 = \frac{2EPh}{A\,l}$$

$$\boxed{f = \sqrt{\frac{2EPh}{A\,l}}}$$

4) Impact by shock

Consider a body subjected to a shock load Let, A = Cross sectional area of the bar,

l = Length of the bar,

δl = Deformation of the bar due to the load

E = Young's modulus of the material of the bar and

f = Instantaneous stress induced in the bar

The strain energy is stored in the bar as kinetic energy.

$$\therefore \text{Shock energy} = \frac{1}{2} mv^2$$

Where, m = Mass of the body, v = Velocity of the body

But strain energy stored = Shock energy

$$\frac{f^2}{2E} \times Al = \frac{1}{2} mv^2$$

By using the above equation, we can find out the instantaneous stress induced in the bar due to shock load.

SOLVED PROBLEMS

STRESS, STRAIN, ELONGATION AND YOUNG'S MODULUS

Example : 4.1 *(Oct.92, Oct.95, Apr.13, Apr.15)*

A circular bar of 20mm diameter and 300mm long carries a tensile load of 30KN. Find the stress, strain and elongation of the bar. Take $E = 2 \times 10^5 N/mm^2$.

Given :

Diameter of the bar, d = 20 mm

Tensile load, $P = 30\ KN = 30 \times 10^3\ N$

Length, l = 300 mm

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

To find : 1) Stress, f 2) Strain, e 3) Elongaltion, ∂l

Solution :

$$\text{Area, } A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 20^2 = 314.159\ mm^2$$

$$\text{Stress, } f = \frac{\text{Load}}{\text{Area}} = \frac{30 \times 10^3}{314.159} = \boxed{95.493\ N/mm^2}$$

$$\text{Strain, } e = \frac{\text{Stress}}{\text{Young's Modulus}} = \frac{f}{E} = \frac{95.493}{2 \times 10^5} = \boxed{4.774 \times 10^{-4}}$$

$$\text{Elongation, } \partial l = e \times l = 4.774 \times 10^{-4} \times 300 = \boxed{0.143\ mm}$$

Result : 1) Stress, $f = 95.493\ N/mm^2$ 2) Strain, $e = 4.774 \times 10^{-4}$
3) Elongation, δl **= 0.143 mm**

Example : 4.2 *(Apr.14)*

A mild steel rod of 25mm diameter and 200mm long is subjected to an axial pull of 75KN. If $E = 2.1 \times 10^5 N/mm^2$, ***determine the elongation of the bar.***

Given :

Diameter of the rod, d = 25 mm

Length, l = 200 mm

Load , $P = 75\ KN = 75 \times 10^3\ N$

Young's modulus, $E = 2.1 \times 10^5\ N/mm^2$

To find : 1) Elongation, ∂l

Solution :

$$\text{Area, } A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 25^2 = 490.873\ mm^2$$

$$\text{Elongation, } \partial l = \frac{P\ l}{A\ E} = \frac{75 \times 10^3 \times 200}{490.873 \times 2.1 \times 10^5} = \boxed{0.1455\ mm}$$

Result : 1) Elongation, δl **= 0.1455 mm**

Example : 4.3 *(Apr.02)*

A rectangular wooden column of length 3m and size 300 × 200mm carries an axial load of 300KN. The column is found to be shortened by 1.5mm under the load. Find the stress and strain.

Given : Length of the column, l = 3 m = 3000 mm

Width, b = 300 mm

Depth, d = 200 mm

Change in length, δl = 1.5 mm

Load , P = 300 KN = 300×10^3 N

To find : 1) Stress, f 2) Strain, e

Solution :

Area, A = b × d = 300 × 200 = 60000 mm^2

$$\text{Stress}, f = \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{300 \times 10^3}{60000} = 5\ N/mm^2$$

$$\text{Strain}, e = \frac{\text{Change in length}}{\text{Original length}} = \frac{\delta l}{l} = \frac{1.5}{3000} = 0.0005$$

Result : 1) Stress, f = 5 **N/mm²** 2) Strain, e = **0.0005**

Example : 4.4 *(Oct.93, Oct.14)*

A brass tube of 50mm outside diameter and 45mm inside diameter and 300mm long is compressed between end washers with a load of 24.5KN. Reduction in length is 0.15mm. Determine the value of E.

Given : External diameter, d_1 = 50 mm

Internal diameter, d_2 = 45 mm

Length, l = 300 mm

Load , P = 24.5 KN = 24.5×10^3 N

Change in length, δl = 0.15 mm

To find : 1) Young's modulus, E

Solution :

$$\text{Area}, A = \frac{\pi}{4}(d_1^2 - d_2^2) = \frac{\pi}{4}(50^2 - 45^2) = 373.064\ mm^2$$

$$\text{We know that}, \delta l = \frac{P\,l}{AE}$$

$$\therefore E = \frac{P\,l}{A\,\delta l} = \frac{24.5 \times 10^3 \times 300}{373.064 \times 0.15} = 1.3135 \times 10^5\ N/mm^2$$

Result : 1) Young's modulus, E = 1.3135×10^5 N/mm^2

Example : 4.5 *(Apr.88)*

A rod of hydraulic lift is 1.2m long and 32mm in diameter. It is attached to a plunger of 100mm in diameter working under a pressure of 8 N/mm^2. If $E = 2 \times 10^5 N/mm^2$, find the change in length of the rod.

Given : Length of the rod, l = 1.2 m = 1200 mm

Diameter of the rod, d = 32 mm

Diameter of the plunger, D = 100 mm

Pressure on the plunger, p = $8 N/mm^2$

Young's modulus, E = $2 \times 10^5 N/mm^2$

To find : 1) Change in length, ðl

Solution :

Area of the plunger $= \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times 100^2 = 7853.982\ mm^2$

Area of the rod, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 32^2 = 804.248\ mm^2$

Load on the rod, P = Force on the plunger

= Pressure × Area of the plunger

$= 8 \times 7853.982 = 62831.856\ N$

Change in length, $\delta l = \frac{P\ l}{AE} = \frac{62831.856 \times 1200}{804.248 \times 0.2 \times 10^6} = 0.469\ mm$

Result : 1) Change in length of the rod, δl **= 0.469 mm**

WORKING STRESS, FACTOR OF SAFETY

Example : 4.6 *(Oct.92, Oct.94, Apr.01, Oct.02, Oct.03, Apr.05)*

A cement concrete cube of 150mm size crushes at a load of 337.5KN. Determine the working stress, if the factor of safety is 3.

Given : Side of the cube, s = 150 mm

Crush load, P = 337.5 KN = 337.7×10^3 N

Factor of safety = 3

To find : 1) Working stress, f_r

Solution :

Area, $A = s^2 = 150 \times 150 = 22500\ mm^2$

Ultimate stress, $f_u = \frac{\text{Crush load}}{\text{Area}} = \frac{P}{A} = \frac{337.5 \times 10^3}{22500} = 15\ N/mm^2$

Factor of safety $= \frac{\text{Ultimate stress}}{\text{Working stress}}$

Working stress, $f_r = \frac{\text{Ultimate stress}}{\text{Factor of safety}} = \frac{15}{3} = $ **5 N/mm²**

Result : The working stress, f_w = **5** N/mm²

Example : 4.7 *(Aor.95)*

A hollow cast iron column 250mm diameter with a wall thickness of 25mm is subjected to an axial load. If the ultimate crushing stress for the material is 480 N/mm², calculate the safe load for the column using a factor of safety of 3.

Given : External diameter, d_1 = 250 mm

Wall thickness, t = 25 mm

Ultimate stress, f_u = 480 N/mm²

Factor of safety = 3

To find : 1) Load, P

Solution :

Internal diameter, $d_2 = d_1 - 2t = 250 - (2 \times 25) = 200$ mm

Area, $A = \frac{\pi}{4}(d_1^2 - d_2^2) = \frac{\pi}{4}(250^2 - 200^2) = 17671.459$ mm²

Working stress, $f_r = \frac{\text{Ultimate stress}}{\text{Factor of safety}} = \frac{480}{3} = 160$ N/mm²

Also, working stress, $f_r = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$

∴ Load, P = Working stress × Area

= 160 × 17671.4590 = **2827433.44 N**

Result : 1) Load, P = **2827433.44 N**

Example : 4.8 *(Apr.96)*

The ultimate stress for a hollow steel column which carries an axial load of 2000KN is 480N/mm². If the external diameter of the column is 200mm, determine the internal diameter. Take factor of safety as 4.

Given : Ultimate stress, f_u = 480 N/mm²

Load, P = 2000 KN = 2000 × 10^3 N

External diameter, d_1 = 200 mm

Factor of safety = 4

To find : 1) The internal diameter, d_2

Solution :

Working stress, $f_r = \frac{\text{Ultimate stress}}{\text{Factor of safety}} = \frac{480}{4} = 120\ N/mm^2$

Also, working stress, $f_r = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$

Area = $\frac{\text{Load}}{\text{Working Stress}} = \frac{2000 \times 10^3}{12}$ = 16666.666 mm^2

Let d_2 be the internal diameter of the column, then

Area, $A = \frac{\pi}{4}(d_1^2 - d_2^2)$

$16666.666 = \frac{\pi}{4} \times (200^2 - d_2^2)$

$21220.662 = 40000 - d_2^2$

$d_2^2 = 18779.338$

$d_2 = \sqrt{18779.338}$ = **137.038 mm**

Result : 1) The internal diameter, d_2 = **137.038 mm**

STRESS – STRAIN DIAGRAM

Example : 4.9 *(Apr.92)*

The following observations were obtained on a mild steel specimen having an initial gauge length of 50mm and initial diameter of 16mm: Load at yield point = 60KN; Maximum load = 88KN; load at fracture = 64KN; Distance between gauge points after fracture = 68.8 mm; Diameter of the neck = 9.2mm. Determine the 1) yield stress, 2) ultimate stress, 3) nominal stress at the fracture, 4) percentage elongation and 5) percentage reduction in area.

Given : Initial diameter, d = 16 mm

Diameter of the neck, d_0 = 9.2 mm

Initial gauge length, l = 50 mm

Distance between gauge points after fracture, l_0 = 68.8 mm

Load at yield point = 60 KN = 60×10^3 N

Maximum load = 88 KN = 88×10^3 N

Load at fracture = 64 KN = 64×10^3 N

To find : 1) Yield stress 2) Ultimate stress

3) Nominal stress at fracture 4) Percentage of elongation

5) Percentage reduction in area

Solution :

Original area of cross section, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 16^2 = 201.06\ mm^2$

Area of neck after fracture, $A_0 = \frac{\pi}{4} \times d_0^{\ 2} = \frac{\pi}{4} \times 9.2^2 = 66.48\ mm^2$

$$\text{Yield stress} = \frac{\text{Load at the yield point}}{\text{Original area of cross section}}$$

$$= \frac{60 \times 10^3}{201.06} = \boxed{298.42\ N/mm^2}$$

$$\text{Ultimate stress} = \frac{\text{Maximum load}}{\text{Original area of cross section}}$$

$$= \frac{88 \times 10^3}{201.06} = \boxed{437.68\ N/mm^2}$$

$$\text{Maximum stress at fracture} = \frac{\text{Load at the fracture}}{\text{Original area of cross section}}$$

$$= \frac{64 \times 10^3}{201.06} = \boxed{318.31\ N/mm^2}$$

$$\text{Pecentage elongation} = \frac{(l_0 - l)}{l} \times 100 = \frac{(68.8 - 50)}{50} \times 100 = \boxed{37.6\%}$$

$$\text{Pecentage reduction in area} = \frac{(A - A_0))}{A} \times 100$$

$$= \frac{(201.06 - 66.48)}{201.06} \times 100 = \boxed{66.94\ \%}$$

Result :
1) Yield stress = **298.42** N/mm^2
2) Ultimate stress = **437.68** N/mm^2
3) Nominal stress at fracture = **318.31** N/mm^2
4) Percentage of elongation = **37.6 %**
5) Percentage reduction in area = **66.94 %**

BARS OF VARYING CROSS SECTIONS

Example : 4.10 *(Oct.92, Oct.04)*

A stepped bar of 1m length is composed of two segments of equal length. The first segment is 20×20mm square and the other is 40×40mm square in size. Calculate the elongation of the bar, when the maximum tensile stress in the material is 200 N/mm^2 ***due to an axial tensile force. Take*** $E = 2 \times 10^5 N/mm^2$.

Given : Area of the first segment, $A_1 = 20 \times 20 = 400\ mm^2$

Area of the second segment, $A_2 = 40 \times 40\ 1600\ mm^2$

Maximum stress in the material, $f = 200\ N/mm^2$

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

Length of the first segment, $l_1 = 500$ mm

Length of the second segment, $l_2 = 500$ mm

To find : 1) Total change in length, δl

Solution :

Maximum tensile stress occurs only in the segments having small area of cross section. So, the stress in the first segment, $f_1 = 200\ N/mm^2$

Load on the material, $P = f_1 \times A_1 = 200 \times 400 = 80000$ N

$$\text{Total change in length, } \delta l = \frac{P\,l_1}{A_1\,E} + \frac{P\,l_2}{A_2\,E}$$

$$= \frac{80000 \times 500}{400 \times 2 \times 10^5} + \frac{80000 \times 500}{1600 \times 2 \times 10^5} = \boxed{0.625\ \text{mm}}$$

Result : 1) Total change in length, δl **= 0.625 mm**

Example : 4.11

(Oct.98)

A steel bar is 500mm long. The two ends are 35mm and 25mm in diameter and each end portion is 150mm long. The middle portion is 200mm long and 20mm in diameter. Calculate the total extension in the bar if it carries an axial pull of 30KN. Take E=200KN/mm².

Given : Load, $P = 30KN = 30 \times 10^3$ N

Diameter of the first portion, $d_1 = 35$ mm

Length of the first portion, $l_1 = 150$ mm

Diameter of the second portion, $d_2 = 20$ mm

Length of the second portion, $l_2 = 200$ mm

Diameter of the third portion, $d_3 = 25$ mm

Length of the third portion, $l_3 = 150$ mm

Young's modulus, $E = 200\ KN/mm^2 = 2 \times 10^5\ N/mm^2$

To find : 1) Total change in length, δl

Solution :

Area of the first portion, $A_1 = \frac{\pi}{4} \times d_1^2 = \frac{\pi}{4} \times 35^2 = 962.113\ mm^2$

Area of the second portion, $A_2 = \frac{\pi}{4} \times d_2^2 = \frac{\pi}{4} \times 20^2 = 314.159\ mm^2$

Area of the third portion, $A_3 = \frac{\pi}{4} \times d_3^2 = \frac{\pi}{4} \times 25^2 = 490.874\ mm^2$

$$\text{Total change in length, } \delta l = \frac{P\,l_1}{A_1\,E} + \frac{P\,l_2}{A_2\,E} + \frac{P\,l_3}{A_3\,E}$$

$$= \frac{P}{E}\left[\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right]$$

$$= \frac{30 \times 10^3}{2 \times 10^5}\left[\frac{150}{962.113} + \frac{200}{314.159} + \frac{150}{490.874}\right]$$

0.1647 mm

Result : 1) Total change in length, δl = **0.1647 mm**

Example : 4.12

(Oct.98)

A steel bar is 450mm long. The two ends are 15mm diameter and have equal lengths. It is subjected to a tensile load of 15KN. If the stress in the middle portion is limited to 160 N/mm^2, determine the diameter of that portion. Find also the length of the middle portion if the total elongation of the bar is 0.25mm. Young's modulus of the material is given as $E = 2 \times 10^5 N/mm^2$.

Given : Total length of the bar, l = 450 mm

Diameter of two end portions, $d_1 = d_2 = 15$ mm

Total load, P = 15 KN = 15×10^3 N

Stress in the middle portion, $f_2 = 160 \ N/mm^2$

Total elongation, $\delta l = 0.25$ mm

Young's modulus, $E = 2 \times 10^5 \ N/mm^2$

To find : 1) Diameter of the middle portion, d_2

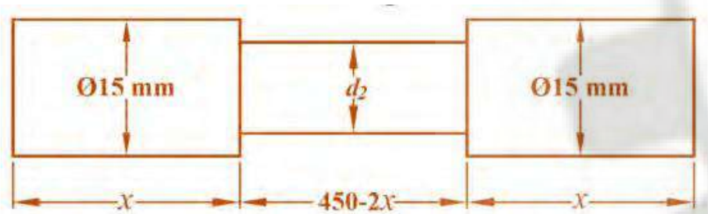


Fig.P4.1 Bar of varying sections [Exapmle 4.12]

Solution :

Let d_2 be the diameter of the middle portion

$$\text{Then, } f_2 = \frac{P}{A_2}$$

$$\therefore A_2 = \frac{P}{f_2} = \frac{15 \times 10^3}{16} = 93.75 \ mm^2$$

$$\text{Also, } A_2 = \frac{\pi}{4} \times d_2^2$$

$$93.75 = \frac{\pi}{4} \times d_2^2$$

$d_2^2 = 119.366$; d_2 = **10.925 mm**

Area of the end portion, $A_1 = A_3 = \frac{\pi}{4} \times d_1^2 = \frac{\pi}{4} \times 15^2 = 176.715\ mm^2$

Let, the length of the end portion, $l_1 = l_3 = x$

Length of the middle portion, $l_2 = 450 - 2x$

Total elongation of the bar, $\delta l = \frac{P}{E}\left[\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right]$

$$0.25 = \frac{15 \times 10^3}{2 \times 10^5}\left[\frac{x}{176.715} + \frac{450 - 2x}{93.75} + \frac{x}{176.715}\right]$$

$$0.25 = 0.075[0.0056588x + 4.8 - 0.0213333x + 0.0056588x]$$

$$3.3333333 = 4.8 - 0.0100157x$$

$$x = \frac{1.4666667}{0.0100157} = 146.437$$

Length of the middle portion, $l_2 = 450 - 2x = 450 - (2 \times 146.437) = $ **157.126 mm**

Result : 1) Diameter of middle portion, d_2 = **10.925 mm**

2) Length of middle portion, l_2 = **157.126 mm**

SHEAR STRESS

Example : 4.13 *(Apr.93)*

A steel punch can be worked on to the compressive stress of 800 N/mm². Find the least diameter of the hole which can be punched through a steel plate 28mm thick if the ultimate shear stress for the plate is 360 N/mm².

Given : Compressive stress on punch, $f = 800\ N/mm^2$

Thickness of steel plate, $t = 23\ mm$

Shear stress, $f_s = 300\ N/mm^2$

To find : 1) Least diameter of hole, d

Solution :

Let the least diameter of the hole = d

Diameter of the punch = Diameter of the hole = d

Compressive force from the punch = Compressive stress × Area of the punch

$$= P \times \frac{\pi}{4} \times d^2 = 800 \times \frac{\pi}{4} \times d^2$$

$$= 628.318\ d^2$$

Resisting force from the plate = Shear stress ×Resisting area of the plate

$= f_s \times \pi dt \quad = 300 \times \pi \times d \times 23$

$= 21676.984\ d$

We know that,

Compressive force from the punch = Resisting force from the plate

$628.318\ d^2 = 21676.984\ d$

$d = \frac{21676.984}{628.318} =$ **34.5 mm**

Result : 1) The least diameter of the hole, d **= 34.5 mm**

LATERAL STRAIN, POISSON'S RATIO, VOLUMETRIC STRAIN, ELASTIC CONSTANTS

Example : 4.14 *(Apr.01, Oct.04, Oct.13, Apr.17)*

A steel bar of 25mm diameter and length of 1m is subjected to a pull of 25KN. If $E = 2 \times 10^5 N/mm^2$, find the elongation, decrease in diameter and increase in volume of the bar. Take 1/m = 0.25.

Given : Diameter of the steel bar, d = 25 mm

Length of the steel bar, l = 1 m = 1000 mm

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

Poison's ratio, 1/m = 0.25

To find : 1) Change in length, δl 2) Change in diameter, δd

3) Change in volume, δV

Solution :

Area of the steel bar, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 25^2 = 490.874\ mm^2$

Volume of the steel bar, $V = A \times l = 490.874 \times 1000 = 490874\ mm^3$

Longitudinal strain, $e = \frac{P}{AE} = \frac{25 \times 10^3}{490.874 \times 2 \times 10^5} = 2.5465 \times 10^{-4}$

Change in length, δl = Longitudinal strain × Length

$= 2.5465 \times 10^{-4} \times 1000 =$ **0.25465 mm**

Poisson's ratio $= \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

Lateral strain = Poisson's ratio × Longitudinal strain

$= 0.25 \times 2.5465 \times 10^{-4} = 6.36625 \times 10^{-5}$

Change in diameter, δd = Lateral strain × Diameter

$= 6.36625 \times 10^{-5} \times 25 =$ **1.5916×10^{-3} mm**

Volumetric strain = $e\left[1 - \frac{2}{m}\right]$

$$= 2.5465 \times 10^{-4}[1 - 2 \times 0.25] = 1.27325 \times 10^{-4}$$

Change in volume, ∂V = Volumetric strain × Volume

$$= 1.27325 \times 10^{-4} \times 490874 = \boxed{62.5\ \text{mm}^3}$$

Result : 1) Change in length, 6l **= 0.25465 mm**

2) Change in diameter, 6d= 1.5916×10^{-3} **mm**

3) Change in volume, 6Y = $62.5\ \text{mm}^3$

Example : 4.15

(Apr.99, Apr.02)

A steel bar of 500mm length, 60mm width and 20mm thickness is subjected to an axial compression of 168KN. Calculate the final dimension and final volume of the bar. The modulus of elasticity of steel is $2.1 \times 10^5 \text{N/mm}^2$ ***and the Poisson's ratio of steel is 0.3.***

Given : Length of the steel bar, l = 500 mm

Width, b = 60 mm

Thickness, t = 20 mm

Axial compressive load, P = 168 KN = 168×10^3 N

Young's modulus, E = $2.1 \times 10^5 \text{N/mm}^2$

Poisson's ratio, 1/m = 0.3

To find :

1) Final length 2) Final width 3) Final thickness 4) Final volume

Solution :

Volume of the bar, V = b × t × l = 60 × 20 × 500 = 600000 mm^3

Area of the bar along the longitudinal direction,

A = b × t = 60 × 20 = 1200 mm^2

Longitudinal strain, $e = \frac{P}{AE} = \frac{168 \times 10^3}{1200 \times 2.1 \times 10^5} = 6.667 \times 10^{-4}$

Change in length, ∂l = Longitudinal strain × Length

$$= 6.667 \times 10^{-4} \times 500 = 0.3333 \text{ mm}$$

Final length = Original length − Change in length ($\because$ *Compression*)

$$= 500 - 0.3333 = \boxed{499.6667\ \text{mm}}$$

Poisson's ratio = $\frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

Lateral strain = Poisson's ratio × Longitudinal strain

$$= 0.3 \times 6.667 \times 10^{-4} = 2 \times 10^{-4}$$

Change in width, δb = Lateral strain × Width

$= 2 \times 10^{-4} \times 60 = 0.012$ mm

Final width = Original width + Change in width (∵ *Width increases*)

$= 60 + 0.012 =$ **60.012 mm**

Change in thickness, δt = Lateral strain × Thickness

$= 2 \times 10^{-4} \times 20 = 0.004$ mm

Final thickness = Original thickness

+ Change in thickness (∵ *Thickness increases*)

$= 20 + 0.004 =$ **20.004 mm**

$$\text{Volumetric strain} = e\left[1 - \frac{2}{m}\right]$$

$$= 6.667 \times 10^{-4}[1 - 2 \times 0.3] = 2.667 \times 10^{-4}$$

Change in volume, δV = Volumetric strain × Volume

$= 6.667 \times 10^{-4} \times 600000 = 160\ mm^3$

Final volume = Original volume

− Change in volume (∵ *Volume decreases*)

$= 600000 - 160 =$ **599840 mm^3**

Result : 1) Final length = **499.6667 mm** 2) Final width = **60.012 mm**
3) Final thickness = **20.004 mm** 4) Final volume = **599840 mm^3**

Example : 4.16

(Oct.01)

A spherical ball of diameter 200mm when subjected to a hydrostatic pressure of 10 N/mm^2 is found to shrink to a ball of 199.7mm. If the Poisson's ratio of the ball is 0.3, find the Young's modulus of the material of the ball.

Given :

Diameter of the spherical ball, d	= 200 mm
Diameter of the ball after shrinking, d_0	= 199.7 mm
Poisson's ratio, 1/m	= 0.3
Hydrostatic pressure	= 10 N/mm^2

To find : 1) Young's modulus, E

Solution: Stress , f = Hydrostatic pressure = 10 N/mm^2

Change in diameter , $\delta d = d - d_0 = 200 - 199.7 = 0.3$ mm

$$\text{Lateral strain} = \frac{\text{Change in diameter}}{\text{Original diameter}} = \frac{0.3}{200} = 0.0015$$

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$\text{Longitudinal strain} = \frac{\text{Lateral strain}}{\text{Poisson's ratio}} = \frac{0.0015}{0.3} = 0.005$$

$$\text{Young's modulus, } E = \frac{\text{Stress}}{\text{Longitudinal strain}} = \frac{10}{0.005} = \boxed{2000 \text{ N/mm}^2}$$

Result : 1) Young's modulus E = **2000** N/mm^2

Example : 4.17 *(Oct.92, Oct.16, Apr.17)*

A circular bar of length 150 mm and diameter of 50mm is subjected to an axial pull of 400KN. The extension in length and contration in diameter were found to be 0.25mm and 0.02mm respectively after loading. Calculate (i) Poisson's ratio (ii) Young's modulus (iii) Modulus of rigidity and (iv) Bulk modulus.

Given : Length of the bar, l = 150 mm

Diameter of the bar, d = 50 mm

Load, P = 400 KN = 400×10^3 N

Change in length, δl = 0.25 mm

Change in diameter, δd = 0.02 mm

To find : 1) Poisson's ratio, 1/m 2) Young's modulus, E 3) Modulus or rigidity, C 4) Bulk modulus, K

Solution : Area of the steel rod, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 50^2 = 1963.495 \text{ mm}^2$

$$\text{Change in length}, \delta l = \frac{P\,l}{A\,E}$$

$$E = \frac{P\,l}{A\,\delta l} = \frac{400 \times 10^3 \times 150}{1963.495 \times 0.25} = \boxed{1.2223 \times 10^5 \text{ N/mm}^2}$$

$$\text{Lateral strain} = \frac{\delta d}{d} = \frac{0.02}{50} = 0.0004$$

$$\text{Longitudinal strain, } e = \frac{\delta l}{l} = \frac{0.25}{150} = 1.6667 \times 10^{-3}$$

$$\text{Poisson's ratio } \frac{1}{m} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{0.0004}{1.6667 \times 10^{-3}} = \boxed{0.24}$$

$$\text{We know that, } E = 2C\left[1 + \frac{1}{m}\right]$$

$$1.2223 \times 10^5 = 2C[1 + 0.24]$$

$$C = \frac{1.2223 \times 10^5}{2 \times 1.24} = \boxed{4.9286 \times 10^4 \text{ N/mm}^2}$$

$$E = 3K\left[1 - \frac{2}{m}\right] = 3K\left[1 - 2 \times \frac{1}{m}\right]$$

$$1.2223 \times 10^5 = 3K[1 - 2 \times 0.24]$$

$$K = \frac{1.2224 \times 10^5}{3 \times 0.52} = \boxed{7.8353 \times 10^4 \text{ N/mm}^2}$$

Result : 1) Poisson's ratio, $1/m$ **= 0.24**

2) Young's modulus, E=1.2223×10^5 N/mm^2

3) Rigidity modulus, C $= 4.9286 \times 10^4$ **N/mm²**

4) Bulk modulus, K $= 7.8353 \times 10^4$ **N/mm²**

Example : 4.18

(Apr.01)

A steel bar of 30mm diameter is subjected to a tensile load of 70KN. Length of the bar is 400mm. Calculate (i)Extension of the bar under the load 70KN (ii)The change in diameter (iii)Bulk modulus if Young's modulus of the material is 200K N/mm^2 ***and 1/m = 0.22.***

Given :

Diameter of the bar, d = 30 mm

Length of the bar, l = 400 mm

Tensile load, P = 70 KN = 70×10^3 N

Poisson's ratio, $1/m = 0.22$

Young's modulus, E = $200 \times 10^3 N/mm^2$

To find : 1) Change in length, ðl 2) Change in diameter, ðd

3) Bulk modulus, K

Solution :

Area of the steel bar, $A = \frac{Л}{4} \times d^2 = \frac{Л}{4} \times 30^2 = 706.858 \text{ mm}^2$

Longitudinal strain, $e = \frac{P}{AE} = \frac{70 \times 10^3}{706.858 \times 200 \times 10^3} = 4.951 \times 10^{-4}$

Change in length , ðl = Longitudinal strain × Length

$= 4.951 \times 10^{-4} \times 400 = \boxed{0.198 \text{ mm}}$

Poisson's ratio, $1/m = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

Lateral strain = Poisson's ration × Longitudinal strain

$= 0.22 \times 4.951 \times 10^{-4} = 1.0892 \times 10^{-4}$

Change in diameter , ðd = Lateral strain × Diameter

$= 1.0892 \times 10^{-4} \times 30 = \boxed{3.2676 \times 10^{-3} \text{mm}}$

We know that, $E = 3K\left[1 - \frac{2}{m}\right] = 3K\left[1 - 2 \times \frac{1}{m}\right]$

$$200 \times 10^3 = 3K[1 - 2 \times 0.22]$$

$$K = \frac{200 \times 10^3}{3 \times 0.56} = \boxed{1.19048 \times 10^5 \text{ N/mm}^2}$$

Result : 1) Change in length, δl **= 0.198 mm**
2) Change in diameter, δd **=** 3.2676×10^{-3} **mm**
3) Bulk modulus, K **=** 1.19048×10^5 **N/mm²**

Example : 4.19

(Apr.94, Apr.03)

For a given material, the Young's modulus is 1×10^5 N/mm² ***and modulus of rigidity is*** 0.4×10^5 N/mm²***. Find the bulk modulus and lateral contraction of a round bar of 50mm diameter and 2.5m long when stretched by 2.5mm.***

Given : Young's modulus, E = 1×10^5 N/mm²
Rigidity modulus, C = 0.4×10^5 N/mm²
Diameter of the bar, d = 50 mm
Length of the bar, l = 2.5 m = 2500 mm
Change in length, ðl = 2.5 mm

To find : 1) Bulk modulus, K 2) Change in diameter, ðd

Solution : We know that, $E = 2C\left[1 + \frac{1}{m}\right]$

$$1 \times 10^5 = 2 \times 0.4 \times 10^5 \left[1 + \frac{1}{m}\right]$$

$$\left[1 + \frac{1}{m}\right] = \frac{1 \times 10^5}{2 \times 0.4 \times 10^5} = 1.25$$

$$\frac{1}{m} = 1.25 - 1 = 0.25$$

$$\text{Also, } E = 3K\left[1 - \frac{2}{m}\right] = 3K\left[1 - 2 \times \frac{1}{m}\right]$$

$$1 \times 10^5 = 3K[1 - 2 \times 0.25]$$

$$K = \frac{1 \times 10^5}{3 \times 0.5} = \boxed{0.667 \times 10^5 \text{ N/mm}^2}$$

$$\text{Longitudinal strain, } e = \frac{\partial l}{l} = \frac{2.5}{2500} = 0.001$$

$$\text{Poisson's ratio, } 1/m = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

Lateral strain = Poisson's ratio × Longitudinal strain

$= 0.25 \times 0.001 = 0.25 \times 10^{-3}$

Change in diameter, ðd = Lateral strain × Diameter $= 0.25 \times 10^{-3} \times 50 =$ **0.0125 mm**

Result : 1) Bulk modulus, K = 0.667×10^5 N/mm^2

2) Change in diameter, δd = **0.0125 mm**

Example : 4.20 *(Apr.90, Oct.91, Apr.04)*

In a tensile test on a hollow tube of external diameter 18mm and internal diameter 12mm, an axial load of 1700N produced an elongation of 0.0045mm in length of 75mm while diameter suffered a compression of 0.00032mm. Calculate the Poisson's ratio, Young's modulus and bulk modulus.

Given : External diameter of the tube, $d_1 = 18$ mm

Internal diameter of the tube, $d_2 = 12$ mm

Axial load, P = 1700 N

Change in length, ðl = 0.0045 mm

Length, l = 75 mm

Change in diameter, ðd = 0.00032 mm

To find : 1) Poisson's ratio, 1/m 2) Young's modulus, E

3) Bulk modulus, K

Solution :

Area of tube, $A = \frac{\pi}{4}(d_1^2 - d_2^2) = \frac{\pi}{4}(18^2 - 12^2) = 141.372\ mm^2$

Lateral strain $= \frac{ðd}{d_1} = \frac{0.00032}{18} = 1.778 \times 10^{-5}$

Longitudinal strain, $e = \frac{ðl}{l} = \frac{0.0045}{75} = 6 \times 10^{-5}$

Poisson ratio, $1/m = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{1.778 \times 10^{-5}}{6 \times 10^{-5}} =$ **0.2963**

Stress, $f = \frac{Load}{Area} = \frac{1700}{141.372} = 12.025\ N/mm^2$

$E = \frac{f}{e} = \frac{12.025}{6 \times 10^{-5}} =$ **2.0042×10^5 N/mm^2**

We know that, $E = 3K\left[1 - \frac{2}{m}\right] = 3K\left[1 - 2 \times \frac{1}{m}\right]$

Young's modulus, E =

$2.0042 \times 10^5 = 3K[1 - 2 \times 0.2963]$

$K = \frac{2.0042 \times 10^5}{3 \times 0.4074} =$ **1.6398×10^5 N/mm^2**

Result :
1) Poisson's ratio, $1/m$ = **0.2963**
2) Young's modulus, $E = 2.0042 \times 10^5$ **N/mm^2**
3) Bulk modulus, $K = 1.6398 \times 10^5$ **N/mm^2**

Example : 4.21

(Oct.94, Oct.17)

A bar of steel 28mm diameter and 250mm long is subjected to an axial load of 80KN. It is found that the diameter has contracted by 1/240mm. If the modulus of rigidity is $0.8 \times 10^5 N/mm^2$, calculate (1) Poisson's ratio (2) Young's modulus and (3) Bulk modulus.

Given :

Diameter, $d = 28$ mm

Length , $l = 250$ mm

Axial load, $P = 80$ KN $= 80 \times 10^3$ N

Change in diameter, $\delta d = 1/240 = 4.1667 \times 10^{-3}$ mm

Modulus of rigidity, $C = 0.8 \times 10^5$ N/mm^2

To find : : 1) Poisson's ratio, $1/m$ 2) Young's modulus, E 3) Bulk modulus, K

Solution :

$$\text{Area},\ A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 28^2 = 615.752\ mm^2$$

$$\text{Lateral strain} = \frac{\delta d}{d} = \frac{4.1667 \times 10^{-3}}{28} = 1.4881 \times 10^{-4}$$

$$\text{Longitudinal strain},\ e = \frac{P}{A\,E} = \frac{80 \times 10^3}{615.752 \times E} = \frac{129.922}{E}$$

$$\text{Poisson's ratio},\ 1/m = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= \frac{1.4881 \times 10^{-4}}{(129.922/E)} = 1.14538 \times 10^{-6} E$$

We know that, $E = 2C\left[1 + \frac{1}{m}\right]$

$$E = 2 \times 0.8 \times 10^5 (1 + 1.14538 \times 10^{-6} E)$$

$$E = 1.6 \times 10^5 + 0.18326E$$

$$(1 - 0.18326)\,E = 1.6 \times 10^5$$

$$E = \frac{1.6 \times 10^5}{0.8167} = \boxed{1.959 \times 10^5\ \text{N/mm}^2}$$

$$\text{Poisson ratio},\ \frac{1}{m} = 1.14538 \times 10^{-6} \times 1.959 \times 10^5 = \boxed{0.2244}$$

Also, $E = 3K\left[1 - \frac{2}{m}\right]$

$1.959 \times 10^5 = 3K[1 - 2 \times 0.2244]$

$K = \frac{1.959 \times 10^5}{3 \times 0.5512} = $ **1.1847×10^5 N/mm²**

Result :
1) Poisson's ratio, $1/m$ = **0.2244**
2) Young's modulus, E = 1.959×10^5 **N/mm²**
3) Bulk modulus, K = 1.1847×10^5 **N/mm²**

COMPOSITE BARS

Example : 4.22 *(Oct.92, Oct.15, Apr.17)*

Two vertical wires each 2.5mm diameter and 5m long jointly support a weight of 2.5KN. One wire is steel and the other is of different material. If the wires stretch elastically 6mm, find the load taken by each and the value of Young's modulus for the second wire if that of steel is 0.2×10^6 N/mm².

Given :

Diameter of the wire, d = 2.5 mm

Length of each wire, l = 5 m = 5000 mm

Elongation of each wire, ðl = 6 mm

Total load, P = 2.5 KN = 2500 N

Young's modulus of steel, $E_1 = 0.2 \times 10^6$ N/mm²

To find : 1) Load taken by each wire P_1 & P_2

2) Young's modulus of the second wire, E_2

Solution :

Area of each wire, $A_1 = A_2 = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 2.5^2 = 4.909$ mm²

We know that, elongation, $\delta l = \frac{P_1 l}{A_1 E_1}$

$P_1 = \frac{A_1 E_1 \delta l}{l} = \frac{4.909 \times 0.2 \times 10^6 \times 6}{5000} = $ **1178.16 N**

Total load = $P_1 + P_2$

$2500 = 1178.16 + P_2$

$P_2 = 2500 - 1178.16 = $ **1321.84 N**

Also elongation, $\delta l = \frac{P_2 l}{A_2 E_2}$

$6 = \frac{1321.84 \times 5000}{4.909 \times E_2}$

$E_2 = \frac{P_2 l}{A_2 \delta l} = \frac{1321.84 \times 5000}{4.909 \times 6} = $ **2.244×10^5 N/mm²**

Result : 1) Load taken by first wire, P_1 = **1178.16 N**
2) Load taken by second wire, P_2 = **1321.84 N**
3) Young's modulus of second wire, $E_2 = 2.244 \times 10^5$ **N/mm^2**

Example : 4.23

(Oct.93, Oct.02)

A solid copper rod 36mm diameter is rigidly fixed at both ends inside a tube of 45mm inside diameter and 50mm outside diameter. The composite section is then subjected to an axial pull of 98KN. Determine the stresses induced in the rod and tube and total elongation of the composite section in length of 1m. E for copper is $1.1 \times 10^5 \text{N/mm}^2$ ***and E for steel is*** $2 \times 10^5 \text{N/mm}^2$.

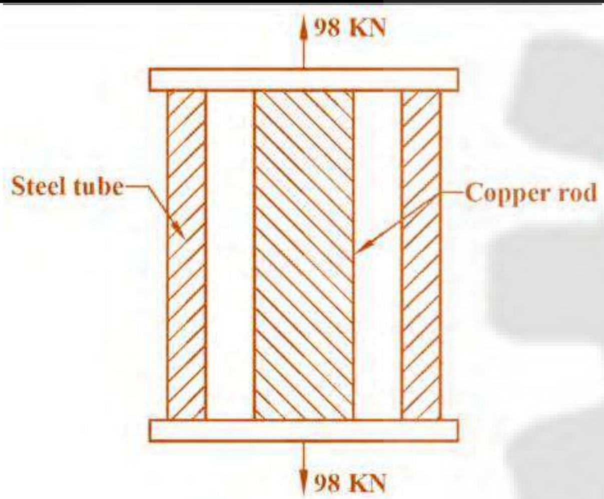


Fig.P4.2 Composite bar [Exapmle 4.23]

Given : Diameter of solid copper rod, d_c = 36 mm
External diameter of steel tube, d_1 = 50 mm Internal diameter of steel tube, d_2 = 45 mm
Axial pull, P = 98 KN = 98×10^3 N
Length of composite section, l = 1 m = 1000 mm
Young's modulus of copper, $E_c = 1.1 \times 10^5 \text{ N/mm}^2$
Young's modulus of steel, $E_s = 2 \times 10^5 \text{ N/mm}^2$

To find :
1) The stress induced in the copper, f_c
2) The stress induced in the steel, f_s
3) Total elongation, ðl

Solution :

Area of copper rod, $A_c = \frac{\pi}{4} \times d_c^{\,2} = \frac{\pi}{4} \times 36^2 = 1017.876 \text{ mm}^2$

Area of steel tube, $A_s = \frac{\pi}{4} \times (d_1^{\,2} - d_2^{\,2}) = \frac{\pi}{4} \times (50^2 - 45^2) = 373.064 \text{ mm}^2$

In a composite bar, the strain per unit length will be same for both the materials.

i.e. $\frac{f_s}{E_s} = \frac{f_c}{E_c}$

$$f_s = \frac{E_s \times f_c}{E_c} = \frac{2 \times 10^5 \times f_c}{1.1 \times 10^5} = 1.818 f_c \quad \text{- - - - - (1)}$$

Total load $= P_s + P_c = f_s A_s + f_c A_c$

$$98000 = 373.064 f_s + 1017.876 f_c \quad \text{- - - - - (2)}$$

Substitute the value of f_s in (2), we get

$$98000 = (373.064 \times 1.818 f_c) + 1017.876 f_c$$

$$98000 = 1696.106 f_c$$

$$f_c = \frac{98000}{1696.106} = \boxed{57.779 \text{ N/mm}^2}$$

Substitute the value of f_c in (1), we get

$$f_s = 1.818 \times 57.779 = \boxed{105.042 \text{ N/mm}^2}$$

$$\text{Total elongation, } \delta l = \frac{f_s\, l}{E_s} \text{ (or) } \frac{f_c\, l}{E_c}$$

$$= \frac{57.779 \times 1000}{1.1 \times 10^5} \text{ (or) } \frac{105.042 \times 1000}{2 \times 10^5}$$

$$= \boxed{0.5253 \text{ mm}}$$

Result : 1) The stress induced in the copper, f_c = **57.779** N/mm^2

2) The stress induced in the steel, f_s = **105.042** N/mm^2

3) Total elongation, 6l = **0.5253 mm**

Example : 4.24

(Oct.13, Apr.15)

A copper rod of 30mm diameter is surrounded tightly by a cast iron tube 60mm external diameter, their ends being firmly fastened together. When they are subjected to a compressive load of 12KN axially, what load is taken by each member? Also determine the contraction of the bar if their length is 100mm originally. The Young's modulus of copper is $0.1 \times 10^6 \text{ N/mm}^2$ ***and that of C.I is*** $0.12 \times 10^6 \text{ N/mm}^2$.

Given : Diameter of the copper rod, $d_c = 30$ mm

External diameter of C.I tube, $d_1 = 60$ mm

Internal diameter of C.I tube, $d_2 = 30$ mm

Total load, $P = 12$ KN $= 12 \times 10^3$ N

Young's modulus of copper, E_c = 0.1×10^6 N/mm^2

Young's modulus of C.I, E_{ci} = 0.12×10^6 N/mm^2

To find : 1) Load taken by the copper rod, P_c

2) Load taken by the C.I tube, P_{ci}

3) Contraction of the bar, δl

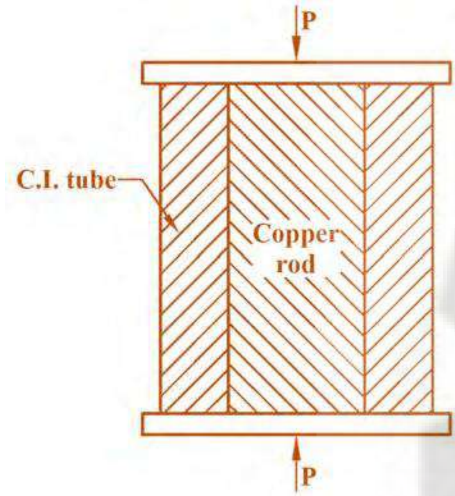


Fig.P4.3 Composite bar [Exapmle 4.24]

Solution :

Area of copper rod, $A_c = \frac{\pi}{4} \times d_c^2 = \frac{\pi}{4} \times 30^2 = 706.858\ \text{mm}^2$

Area of CI tube, $A_{ci} = \frac{\pi}{4} \times (d_1^2 - d_2^2) = \frac{\pi}{4} \times (60^2 - 30^2) = 2120.575\ \text{mm}^2$

In this composite bar,

Load taken by the copper rod, $P_c = \frac{P \times A_c E_c}{A_c E_c + A_{ci} E_{ci}}$

$$= \frac{12 \times 10^3 \times 706.858 \times 0.1 \times 10^6}{(706.858 \times 0.1 \times 10^6) + (2120.575 \times 0.12 \times 10^6)} = \boxed{2608.695\ \mathbf{N}}$$

Total load , $P = P_c + P_{ci}$

$12 \times 10^3 = 2608.695 + P_{ci}$

Load taken by the CI tube, $P_{ci} = 12 \times 10^3 - 2608.695 = \boxed{9391.305\ \mathbf{N}}$

Contraction of the bar, $\delta l = \frac{P_c l}{A_c E_c} = \frac{2608.695 \times 100}{706.858 \times 1 \times 10^5} = \boxed{\mathbf{3.691 \times 10^{-3}\ mm}}$

Result : 1) Load taken by the copper rod, P_c = **2608.695N**

2) Load taken by the C.I tube, P_{ci} = **9391.305N**

3) Contraction of the bar, δl = **3.691×10^{-3} mm**

Example : 4.25 *(Apr.92)*

A tube of aluminium 40mm external diameter and 20mm internal diameter is snugly fitted on to a steel rod of 20mm diameter. The composite bar is loaded in compression by an axial load P. Find the stress in aluminium when the load is such that the stress in steel rod is 70 N/mm^2***. What is the value of P, if E for steel is*** $2 \times 10^5\ N/mm^2$ ***and E for aluminium is*** $0.7 \times 10^5\ N/mm^2$***.***

Given : Diameter of the steel rod, $d_s = 20$ mm
External diameter of aluminium tube, $d_1 = 40$ mm
Internal diameter of aluminium tube, $d_2 = 20$ mm
Stress induced in steel rod, $f_s = 70\ N/mm^2$
Young's modulus of steel, $E_s = 2 \times 10^5\ N/mm^2$
Young's modulus of aluminium, $E_a = 0.7 \times 10^5\ N/mm^2$

To find : 1) The stress induced in aluminium tube, f_a
2) The total axial load, P

Solution :

Area of steel rod, $A_c = \frac{\pi}{4} \times d_s^2 = \frac{\pi}{4} \times 20^2 = 314.159\ mm^2$

Area of aluminium tube, $A_a = \frac{\pi}{4} \times (d_1^2 - d_2^2) = \frac{\pi}{4} \times (40^2 - 20^2) = 942.478\ mm^2$

In a composite bar, the strain per unit length will be same for both the materials.

i.e. $\frac{f_s}{E_s} = \frac{f_a}{E_a}$

$f_a = \frac{E_a \times f_s}{E_s} = \frac{0.7 \times 10^5 \times 70}{2} = $ **24.5 N/mm²**

Total load, $P = P_s + P_a = f_s A_s + f_a A_a$

$= (70 \times 314.159) + (24.5 \times 942.478) = $ **45081.841 N**

Result : 1) The stress induced in aluminium tube, f_a = **24.5 N/mm²**
2) The total axial load, P = **45081.841N**

Example : 4.26 *(Oct.95, Apr.14)*

A steel tube 100mm internal diameter and 12.5mm thick is surrounded by a brass tube of the same thickness in such a way that the axes of the two tubes coincide. The compound tube is loaded by an axial compressive load of 5KN. Determine the load carried by each tube, the stresses and strain developed in each tube. Assume that there is no buckling of the tubes. Take Young's modulus for steel as $2 \times 10^5\ N/mm^2$ ***and that for brass as*** $1 \times 10^5\ N/mm^2$***. The tubes are of the same length.***

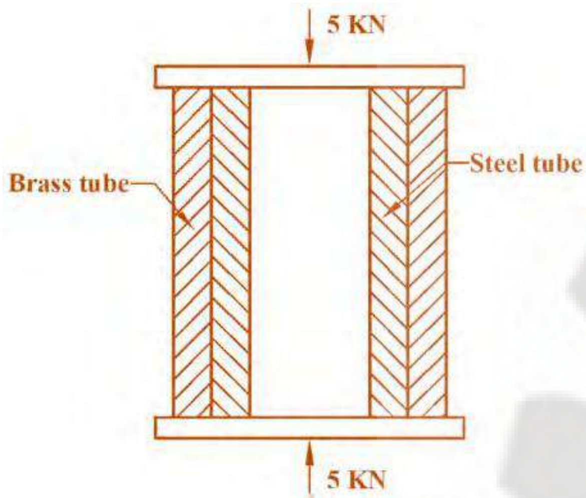


Fig.P4.4 Composite bar [Exapmle 4.26]

Given : Internal diameter of the steel tube, $d_2 = 100$ mm

Thickness, $t = 12.5$ mm

Load, $P = 5$ KN $= 5000$ N

Young's modulus of steel, $E_s = 2 \times 10^5$ N/mm^2

Young's modulus of brass, $E_b = 1 \times 10^5$ N/mm^2

To find : 1) Load carried by the steel tube, P_s

2) Load carried by the brass tube, P_b
3) Stress in steel tube, f_s
4) Stress in brass tube, f_b
5) Strain developed in each tube, e_s or e_b

Solution :

External diameter of steel tube, $d_1 = d_2 + 2t = 100 + (2 \times 12.5) = 125$ mm

Internal diameter of brass tube, $D_2 = d_1 = 125$ mm

External diameter of brass tube, $D_1 = D_2 + 2t = 125 + (2 \times 12.5) = 150$ mm

Area of steel tube, $A_s = \frac{\pi}{4} \times (d_1^2 - d_2^2) = \frac{\pi}{4} \times (125^2 - 100^2) = 4417.865$ mm^2

Area of brass tube, $A_b = \frac{\pi}{4} \times (D_1^2 - D_2^2) = \frac{\pi}{4} \times (150^2 - 125^2) = 5399.612$ mm^2

In this composite bar,

Stress induced in steel rod, $f_s = \dfrac{P \times E_s}{E_s A_s + E_b A_b}$

$$= \frac{5000 \times 2 \times 10^5}{(2 \times 10^5 \times 4417.865) + (1 \times 10^5 \times 5399.612)} = \boxed{0.7024} \text{ N/mm}^2$$

Stress induced in brass tube, $f_b = \dfrac{P \times E_b}{E_s A_s + E_b A_b}$

$$= \frac{5000 \times 1 \times 10^5}{(2 \times 10^5 \times 4417.865) + (1 \times 10^5 \times 5399.612)} = \boxed{0.3512} \text{ N/mm}^2$$

Load carried by steel tube, $P_s = f_s A_s = 0.7024 \times 4417.865 = \boxed{3103.108}$ **N**

Load carried by brass tube, $P_b = P - P_s = 5000 - 3103.108 =$ **1896.892 N**

Stress developed in each tube, $e_s = \dfrac{f_s}{E_s}$ (or) $\dfrac{f_b}{E_b}$

$$= \frac{0.7024}{2 \times 10^5} \text{ (or) } \frac{0.3512}{1 \times 10^5} = \boxed{3.512 \times 10^{-6}}$$

Result : 1) Load carried by the steel tube, P_s **= 3103.108 N**
2) Load carried by the brass tube, P_b **= 1896.892 N**
3) Stress in steel tube, f_s **= 0.7024 N/mm^2**
4) Stress in brass tube, f_b **= 0.3512 N/mm^2**
5) Strain developed in each tube, $e_s = e_b = 3.512 \times 10^{-6}$

Example : 4.27

(Oct.96)

A RCC column 300mm × 450mm has 4 number of 25mm steel rods. Calculate the safe load for the column, if the allowable stress in concrete is 5 N/mm^2 and E for steel is 15 times of E of concrete.

Given : Size of the column = 300 mm × 450 mm

Diameter of one steel rod, d_s = 25 mm

Number of steel rods = 4

Stress in concrete, f_c = 5 N/mm^2

Young's modulus of steel, $E_s = 15\ E_c$

To find : 1) The safe load for the column, P

Solution :

Area of the column = 300 × 450 = 135000 mm^2

Area of one steel rod $= \dfrac{\pi}{4} \times d_s^2 = \dfrac{\pi}{4} \times 25^2 = 490.874$ mm^2

Area of one 4 steel rods = 4 × 490.874 = 1963.496 mm^2

Area of concrete, A_c = Area of column – Area of steel rods

= 135000 – 1963.496 = 133036.51 mm^2

In a composite bar, the strain per unit length will be same for both the materials.

i.e. $\frac{f_s}{E_s} = \frac{f_c}{E_c}$ ⇒ $f_s = \frac{15 \times E_c}{E_c} \times f_c$

$f_s = 15 \times f_c = 15 \times 5 = 75\ N/mm^2$

Load taken by steel rods, $P_s = f_s A_s = 75 \times 1963.496 = 147262.20\ N$

Load taken by concrete, $P_c = f_c A_c = 5 \times 133036.51 = 665182.55\ N$

Total safe load for the column, $P = P_s + P_c$

$= 147262.20 + 665182.55 = 812444.75\ N$ = **812.445 KN**

Result : 1) The safe load for the column, P = **812.445 KN**

Example : 4.28 *(Apr.01)*

A cast iron of 200mm external diameter and 150mm internal diameter is filled with concrete. Determine the stress in cast iron and concrete when an axial compressive load of 50KN is applied. Take E for cast iron = 18 times of E for concrete.

Given : External diameter of C.I tube, $d_1 = 200$ mm

Internal diameter of C.I tube, $d_2 = 150$ mm

Total load, $P = 50\ KN = 50 \times 10^3\ N$

Young's modulus of C.I, $E_{ci} = 18\ E_c$

To find : 1) Stress in cast iron tube, f_{ci} 2) Stress in concrete, f_c

Solution :

Diameter of the concrete, $d_c = d_2 = 150$ mm

Area of concrete, $A_c = \frac{\pi}{4} \times d_c^2 = \frac{\pi}{4} \times 150^2 = 17671.459\ mm^2$

Area of CI tube, $A_{ci} = \frac{\pi}{4} \times (d_1^2 - d_2^2)$

$= \frac{\pi}{4} \times (200^2 - 150^2) = 13744.468\ mm^2$

In a composite bar, the strain per unit length will be same for both the materials.

i.e. $\frac{f_c}{E_c} = \frac{f_{ci}}{E_{ci}}$ ⇒ $\frac{f_{ci}}{18\ E_c} = \frac{f_c}{E_c}$

$f_{ci} = 18 \times f_c$

Total load , $P = P_c + P_{ci}$

$$P = f_c \times A_c + f_{ci} \times A_{ci}$$

$$50 \times 10^3 = (f_c \times 17671.459) + (18 f_c \times 13744.468)$$

$$50 \times 10^3 = 265071.883 f_c$$

$$f_c = \frac{50 \times 10^3}{265071.883} = 0.18863 \text{ N/mm}^2$$

$$f_{ci} = 18 \times f_c = 18 \times 0.18863 = 3.39534 \text{ N/mm}^2$$

Result : 1) The stress in cast iron tube, f_{ci} = 3.39534 N/mm^2

2) The stress in concrete, f_c = 0.18863 N/mm^2

TEMPERATURE STRESSES

Example : 4.29 *(Apr.92)*

Two parallel walls 6 m apart are stayed together by a steel rod 20mm diameter passing through metal plates and nuts at each end. The nuts are tightened when the rod is at a temperature 100°C. Determine the stress in the rod when temperature falls down to 20°C, if (i) the ends do not yield (ii) the ends yield by 1mm. Take $E = 2 \times 10^5 \text{N/mm}^2$ ***and*** $a = 12 \times 10^{-6}$***/°C. Find also the force exerted in both casees.***

Given : Length of the steel rod, l = 6m = 6000 mm

Diameter of the steel rod, d = 20 mm

Initial temperature, T_1 = 100°C

Final temperature, T_2 = 20°C

Amount of yield, ß = 1 mm

Young's modulus, $E = 2 \times 10^5$ N/mm^2

Co–efficient of linear expansion, $\alpha = 12 \times 10^{-6}$ / °C

To find : 1) The stress when the ends do not yield

2) The force exerted when the ends do not yield

3) The stress when the ends yield by 1 mm

4) The force exerted when the ends yield by 1 mm

Solution :

Area of the rod, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 20^2 = 314.159 \text{ mm}^2$

Fall in temperature, $T = T_1 - T_2 = 100 - 20 = 80°C$

The free expansion is prevented when the supports do not yield.

So, temperature stress, $f = \alpha T E$

$$= 12 \times 10^{-6} \times 80 \times 2 \times 10^5 = 192 \text{ N/mm}^2$$

Force exerted, $P = f \times A = 192 \times 314.159 =$ **60318.528 N**

When the supports yield by 1 mm,

Temperature stress, $f = \left[\alpha T - \frac{\delta}{l}\right] E$

$= \left[12 \times 10^{-6} \times 80 - \frac{1}{6000}\right] \times 2 \times 10^5 =$ **158.667 N/mm²**

Force exerted, $F = f \times A = 158.667 \times 314.159 =$ **49846.666 N**

Result :
1) The stress when the ends do not yield = 192 N/mm²
2) The force exerted when the ends do not yield = 60318.528 N
3) The stress when the ends yield by 1mm = 158.667 N/mm²
4) The force exerted when the ends yield by 1mm = 49846.666 N

Example 4.30

(Apr.93)

A railway is laid so that there is no stress in the rail at 50°C. Calculate (i) the expansion allowance for no stress in the rail when the temperature is 150°C (ii) the maximum temperature to have no stress in the rail if the expansion allowance is 26mm per rail. Take $\alpha = 12 \times 10^{-6}$/°C ***and*** $E = 2 \times 10^5$ N/mm². ***The length of the rails is 30m.***

Given :

Initial temperature, $T_1 = 50°C$

Final temperature, $T_2 = 150°C$

Young's modulus, $E = 2 \times 10^5$ N/mm²

Co-efficient of linear expansion, $\alpha = 12 \times 10^{-6}$/°C

Length of the rails, $l = 30\ m = 30 \times 10^3$ mm

Solution :

Rise in temperature, $T = T_2 - T_1 = 150 - 50 = 100°C$

(i) To find the expansion allowance for no stress in the rail

Let δ be the expansion allowance

When there is no stress in the rails, temperature stress = 0

$\left[\alpha T - \frac{\delta}{l}\right] E = 0$

$\left[12 \times 10^{-6} \times 100 - \frac{\delta}{30 \times 10^3}\right] \times 2 \times 10^5 = 0$

$36 - \delta = 0$

$\delta =$ **36 mm**

(ii) To find the maximum temperature to have no stress in the rails, if δ = 26mm

When there is no stress in the rails, temperature stress = 0

$$\left[\alpha T - \frac{ß}{l}\right] E = 0$$

$$\left[12 \times 10^{-6} \times T - \frac{26}{30 \times 10^3}\right] \times 2 \times 10^5 = 0$$

$$0.36\,T - 26 = 0$$

$$T = \frac{26}{0.36} = 72.222°C$$

Maximum temperature = Rise in temperature + Initial temperature

= 72.222 + 50 = **122.222°C**

Result : 1) The expansion allowance required for no stress in the rails when the temperature is 150°C = **36 mm**

2) The maximum temperature to have no stress in the rails, if ß is 26mm = **122.222°C**

STRAIN ENERGY, RESILIENCE & TYPES OF LOADING

Example : 4.31 *(Apr.88, Apr.97, Apr.04, Apr.15, Apr.17)*

Calculate the strain energy that can be stored in a steel bar 70mm in diameter and 6m long, subjected to a pull of 200KN. Assume E=200 KN/mm².

Given : Diameter of the steel bar, d = 70 mm

Length of the steel bar, l = 6 m = 6000 mm

Load, P = 200 KN = 200×10^3 N

Young's modulus, E = 200 KN/mm² = 2×10^5 N/mm²

To find : 1) The strain energy, U

Solution :

Area of rod, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 70^2 = 3848.45 \text{ mm}^2$

Volume of rod, $V = A \times l = 3848.45 \times 6000 = 2.30907 \times 10^7 \text{ mm}^3$

Instantaneous stress, $f = \frac{P}{A} = \frac{200 \times 10^3}{3848.45} = 51.969 \text{ N/mm}^2$

Strain energy, $U = \frac{f^2}{2E} \times \text{Volume}$

$= \frac{51.969^2}{2 \times 2 \times 10^5} \times 2.30907 \times 10^7$

= **155907 N-mm**

Result : 1) The strain energy, U = **155907 N–mm**

Example : 4.32

Calculate the modulus of resilience at a point in a material subjected to a stress of 200 N/mm^2***. Take*** $E = 0.1 \times 10^6\ N/mm^2$***.***

Given : Maximum stress, $f_{max} = 200\ N/mm^2$

Young's modulus, $E = 0.1 \times 10^6\ N/mm^2$

To find : 1) Modulus of resilience

Solution :

$$\text{Modulus of resilience} = \frac{f_{max}^2}{2E} = \frac{200^2}{2 \times 0.1 \times 10^6} = 0.2\ N/mm^2$$

Result : 1) Modulus of resilience = **0.2** N/mm^2

Example : 4.33

(Oct.89, Apr.94, Oct.97, Oct.02,Oct.03)

A steel specimen 150 mm^2 ***cross section stretches by 0.05mm over a 50mm gauge length under an axial load of 30KN. Calculate the strain energy stored in the specimen at this stage, if the load at the elastic limit for the specimen is 50KN. Calculate the elongation at elastic limit and the proof resilience.***

Given : Area of cross section, $A = 150\ mm^2$

Change in length, $\delta l = 0.05\ mm$

Gauge length, $l = 50\ mm$

Axial load, $P = 30\ KN = 30 \times 10^3\ N$

Load at elastic limit, $P_e = 50\ KN = 50 \times 10^3\ N$

To find : 1) Strain energy, U 2) Elongation, δl 3) Proof resilience

Solution :

Volume, $V = A \times l = 150 \times 50 = 7500\ mm^3$

Assume the rod is subjected to gradually applied load.

$$\text{Instantaneous stress, } f = \frac{\text{Axial load}}{\text{Area}} = \frac{30 \times 10^3}{150} = 200\ N/mm^2$$

$$\text{Longitudinal strain, } e = \frac{\text{Change in length}}{\text{Original length}} = \frac{0.05}{50} = 1 \times 10^{-3}$$

$$\text{Young's modulus, } E = \frac{\text{Stress}}{\text{Longitudinal strain}} = \frac{200}{1 \times 10^{-3}} = 2 \times 10^5\ N/mm^2$$

$$\text{Strain energy stored, } U = \frac{f^2}{2E} \times \text{Volume} = \frac{200^2}{2 \times 2 \times 10^5} \times 7500 = 750\ \text{N-mm}$$

Maximum instantaneous stress, $f_{max} = \frac{\text{Load at elastic limit}}{\text{Area}} = \frac{50 \times 10^3}{150} = 333.333 \ N/mm^2$

Proof resilience $= \frac{f_{max}^2}{2 \times E} \times \text{Volume} = \frac{333.333^2}{2 \times 2 \times 10^5} \times 7500 =$ **2083.329 N-mm**

Elongation, $\delta l = \frac{f_{max} \times l}{E} = \frac{333.333 \times 50}{2 \times 10^5} =$ **0.0833 mm**

Result :
1) Strain energy stored, U = **750 N–mm**
2) Elongation at elastic limit, δl = **0.0833 mm**
3) Proof resilience = **2083.329 N–mm**

Example : 4.34

(Oct.04)

A mild steel bar of 10mm diameter and 2m long is subjected to an axial tensile load of 25KN applied suddenly. Find the stress induced and the strain energy stored in the bar. Take $E = 2 \times 10^5 N/mm^2$.

Given :
Diameter of the bar, d = 10 mm
Length of the bar, l = 2 m = 2000 mm
Load, P = 25 KN = 25×10^3 N
Young's modulus, E = $2 \times 10^5 \ N/mm^2$

To find : 1) Stress induced, f 2) Strain energy stored, U

Solution : Area of the rod, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 10^2 = 78.540 \ mm^2$

Volume, $V = A \times l = 78.540 \times 2000 = 157080 \ mm^3$

For suddenly applied load,

Instantaneous stress, $f = 2 \times \frac{P}{A} = 2 \times \frac{25 \times 10^3}{78.540} =$ **636.618 N/mm²**

Strain energy stored, $U = \frac{f^2}{2 \times E} \times \text{Volume}$

$= \frac{636.618^2}{2 \times 2 \times 10^5} \times 157080 =$ **159154.429 N-mm**

Result : 1) Stress induced in the rod, f = **636.618** N/mm^2
2) Strain energy stored, U = **159154.429 N–mm**

Example : 4.35 *(Oct.04 Apr.91, Oct.95, Oct.04, Apr.05)*

Determine the greatest weight that can be dropped from a height of 200mm on to a collar at the lower end of a vertical bar 20mm diameter and 2.5m long without exceeding the elastic limit stress 300 N/mm^2***. Calculate also the instantaneous elongation. Take*** $E = 2 \times 10^5 N/mm^2$.

Given : Height, $h = 200$ mm

Diameter of the bar, d = 20 mm

Length of the bar, l = 2.5 m = 2500 mm

Instantaneous stress, $f = 300\ N/mm^2$

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

To find : 1) The greatest weight that can be dropped, P

2) Elongation, ðl

Solution :

Area of the bar, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 20^2 = 314.159\ mm^2$

Volume, $V = A \times l = 314.159 \times 2500 = 785397.5\ mm^3$

Instantaneous elongation, $\delta l = \frac{f\ l}{E} = \frac{300 \times 2500}{2 \times 10^5} =$ **3.75 mm**

Work done by the load, $W = P\ (h + \delta l) = P\ (200 + 3.75) = 203.75\ P$

Strain energy stored in the bar, $U = \frac{f^2}{2 \times E} \times$ Volume

$= \frac{300^2}{2 \times 2 \times 10^5} \times 785397.5 = 176714.438$ N-mm

Work done = Strain energy stored

$203.75\ P = 176714.438$

$P = \frac{176714.438}{203.75} =$ **867.31 N**

Result : 1) The greatest weight that can be dropped, P = **867.31 N**

2) Elongation, δl = **3.75 mm**

Example : 4.36 *(Oct.91)*

A load of 100N falls by gravity through a vertical distance of 3m, when it is suddenly stopped by a collar at the end of a vertical rod of length 6m and diameter 20mm. The top of the bar is rigidly fixed to a ceiling. Calculate the maximum stress and strain induced in the bar. Take $E = 1.96 \times 10^5 N/mm^2$.

Given : Falling weight, P = 100 N

Height of fall, h = 3 m = 3000 mm

Length of the rod, l = 6m = 6000 mm

Diameter of the rod, d = 20 mm

Young's modulus, E = $1.96 \times 10^5 N/mm^2$

To find : 1) The maximum stress, f 2) Strain, e

Solution :

Area of the rod, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 20^2 = 314.159\ mm^2$

Instantaneous stress, $p = \frac{P}{A} + \sqrt{\frac{P^2}{A^2} + \frac{2EPh}{Al}}$

$= \frac{100}{314.159} + \sqrt{\frac{100^2}{314.159^2} + \frac{2 \times 1.96 \times 10^5 \times 100 \times 3000}{314.159 \times 6000}}$

$= 0.318 + 249.778 =$ **250.096 N/mm²**

Instantaneous strain, $e = \frac{f}{E} = \frac{250.096}{1.96 \times 10^5} =$ 1.276×10^{-3}

Result : 1) The instantaneous stress, f = **250.096** N/mm^2

2) The Instantaneous strain, e = 1.276×10^{-3}

Example : 4.37

(Apr.93, Apr.13, Oct.16)

A weight of 1400N is dropped on to a collar at the lower end of a vertical bar 3m long and 25mm in diameter. Calculate the height of drop, if the maximum instantaneous stress is not to exceed 120 N/mm^2. What is the corresponding instantaneous elongation. Take $E = 2 \times 10^5 N/mm^2$.

Given : Falling weight, P = 1400 N

Length of the bar, l = 3 m = 3000 mm

Diameter of the bar, d = 25 mm

Instantaneous stress, f = 120 N/mm^2

Young's modulus, E = $2 \times 10^5\ N/mm^2$

To find : 1) The height of drop, h 2) Eelongation, δl

Solution :

Area of the bar, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 25^2 = 490.874\ mm^2$

Volume, V = A × l = 490.874 × 3000 = 1472622 mm^3

Elongation, $\delta l = \frac{f\ l}{E} = \frac{120 \times 3000}{2 \times 10^5} =$ **1.8 mm**

Strain energy stored in the bar, U = $\frac{f^2}{2 \times E} \times$ Volume

$$= \frac{120^2}{2 \times 2 \times 10^5} \times 1472622 = 53014.392 \text{ N-mm}$$

Work done by the falling weight = P $(h + \delta l)$ = $1400(h + 1.8)$

Work done = Strain energy stored

$$1400(h + 1.8) = 53014.392$$

$$h + 1.8 = \frac{53014.392}{1400} = 37.8674$$

$$h = 37.8674 - 1.8 = \boxed{36.0674 \text{ mm}}$$

Result : 1) The height of drop, h **= 36.0674 mm**
2) The instantaneous elongation, δl **= 1.8 mm**

Example : 4.38

(Oct.92, Apr.01)

It is found that a bar of 36mm in diameter stretches 2mm under a gradually applied load of 150KN. If a weight of 15KN is dropped on to a collar at the lower end of this bar through a height of 60mm. Calculate the maximum instantaneous stress and elongation produced. Assume E = 215 KN/mm^2.

Given : Diameter of the bar, d = 36 mm

Gradually applied load, P_1 = 150 KN = 150×10^3 N

Elongation under gradually applied load = 2 mm

Falling weight, P = 15 KN = 150000 N

Height of fall of weight, h = 60 mm

Young's modulus, E = $215 KN/mm^2 = 2.15 \times 10^5 N/mm^2$

To find : 1) The maximum instantaneous stress, f

2) The maximum elongation, δl

Solution :

Area of the bar, A = $\frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 36 = 1017.876 \text{ mm}^2$

Elongation under gradually applied load = $\frac{P_1 l}{A E}$

$$2 = \frac{150 \times 10^3 \times l}{1017.876 \times 2.15 \times 10^5}$$

$$l = \frac{2 \times 1017.876 \times 2.15 \times 10^5}{150 \times 10^3} = 2917.911 \text{ mm}$$

Maximum instantaneous stress due to falling weight,

$$f = \frac{P}{A} + \sqrt{\frac{P^2}{A^2} + \frac{2EPh}{Al}}$$

$$= \frac{15000}{1017.876} + \sqrt{\frac{15000^2}{1017.876^2} + \frac{2 \times 2.15 \times 10^5 \times 15000 \times 60}{1017.876 \times 2917.911}}$$

$= 14.7366 + 361.2714 =$ **376.008 N/mm²**

Maximum elongation, $\delta l = \frac{f\,l}{E} = \frac{376.008 \times 2917.911}{2.15 \times 10^5} =$ **5.103 mm**

Result : 1) The maximum instantaneous stress, f = 376.008 **N/mm²**
2) The maximum elongation, δl **= 5.103 mm**

Example : 4.39 *(Apr.01)*

A coach weighing 20KN (is attached to a rope) is traveling down a slope at a speed of 2m/s. It is stopped suddenly by pulling the rope. What is the instantaneous stress and the maximum tension induced in the rope due to sudden stoppage. Assume the length and cross sectional area of the rope to be 100m and 1000 mm² respectively. Take $E = 2 \times 10^5 N/mm^2$.

Given : Weight of the coach, W = 20 KN = 20×10^3 N
Speed of the coach, u = 2 m/s = 2000 mm/s
Length of the rope, l = 100 m = 100×10^3 mm
Area of the rope, A = 1000 mm²
Young's modulus, E = 2×10^5 N/mm²

To find : 1) The maximum instantaneous stress in the rope, f
2) The maximum tension induced in the rope, T

Solution :

When the coach is suddenly stopped, the kinetic energy of the coach is converted into strain energy of the rope.

i.e. $\frac{m u^2}{2} = \frac{f^2}{2E} \times \text{Volume}$

$$\frac{W u^2}{2g} = \frac{f^2}{2E} \times A \times l \quad \left(\because m = \frac{W}{g}\right)$$

$$\frac{20 \times 10^3 \times 2000^2}{2 \times 9.81 \times 10^3} = \frac{f^2 \times 1000 \times 100 \times 10^3}{2 \times 2 \times 10^5} \quad (\because g = 9.81 \times 10^3 \text{ mm/s}^2)$$

$$f^2 = \frac{2 \times 2 \times 10^5 \times 20 \times 10^3 \times 2000^2}{2 \times 9.81 \times 10^3 \times 1000 \times 100 \times 10^3} = 16309.89$$

$f =$ **127.71 N/mm²**

Maximum tension, T = Maximum stress × Area

$= 127.71 \times 1000 = 127710$ N = **127. 71 KN**

Result : 1) The maximum instantaneous stress, f **=127.71** N/mm^2

2) The maximum tension induced in the rope, T **= 127.71 KN**

Unit – III

Chapter 5. GEOMETRICAL PROPERTIES OF SECTIONS

1. Centre of gravity

The centre of gravity of a body may be defined as *a point through which the entire weight of the body is assumed to be concentrated.* It may be noted that every body has only one centre of gravity. It is a term related with
a body having volume and mass i.e. solids.

1. Centroid

The centroid of a section may be defined as *a point through which the entire area of the section is assumed to be concentrated.* It is the term
related with plane figures like rectangle, circle, triangle, etc. having only area but f a plane
figure is

1. Cent

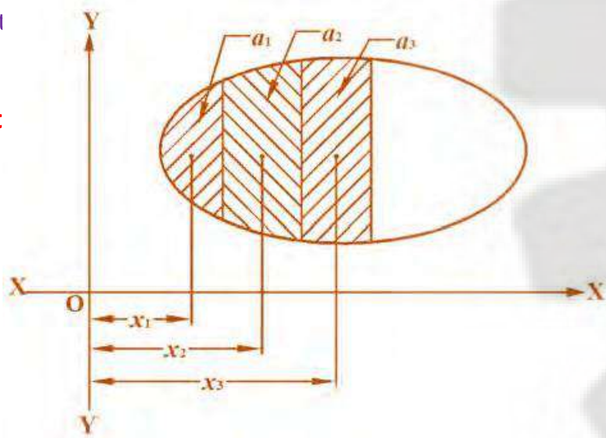


Fig. 5.1 Centroid of a plane figure

Consider a plane figure of area A whose centroid is required to be found out. Divide the plane area into number of small vertical strips as shown in fig.5.1.

Let a_1, a_2, a_3, etc. be the area of the strips and (x_1, y_1), (x_2, y_2), (x_3, y_3), etc. be their co–ordinates of their centroids from a fixed point O. Let, $\bar{X}$ and $\bar{Y}$ be the co–ordinates of the centroid of the plane figure.

Taking moment about Y–Y axis,

The moment of area of first strip $=a_1x_1$

Sum of the moment of areas of all such strips about Y–Y axis.

$$\Sigma ax = a_1x_1 + a_2x_2 + \cdots$$

The moment of area of the whole plane figure about Y–Y axis = $A\bar{X}$

By the principle of moment, $A\bar{X} = \Sigma ax$

$$\bar{X} = \frac{\Sigma ax}{A}$$

$$\bar{X} = \frac{a_1z_1 + a_2z_2 + a_3z_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$$

Similarly,

$$\bar{Y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$$

Centroidal axis

A line passing through the centroid of the plane figure is known as *centroidal axis.*

Axis of reference

A line about which the co–ordinates of centroid are calculated is known as *axis of reference* or *reference axis*.

For plane figures, the axis of reference is taken as lowermost or uppermost line of the figure for calculating $\bar{Y}$ and left extreme line or right extreme line of the figure for calculating $\bar{X}$.

Axis of symmetry

The axis which divides a section into two equal halves horizontally or vertically is known as *axis of symmetry*. The centroid of the section will lie on this axis of symmetry.

5.4 Moment of inertia

The moment of inertia of a body about an axis is defined as the internal resistance offered by the body against the rotation about that axis.

The moment of inertia of a plane figure or lamina about an axis is the product of its area and square of its distance form that axis.

Mathematically, moment of inertia, $I = \Sigma a.\ r^2$

5.5 Moment of inertia a plane figure

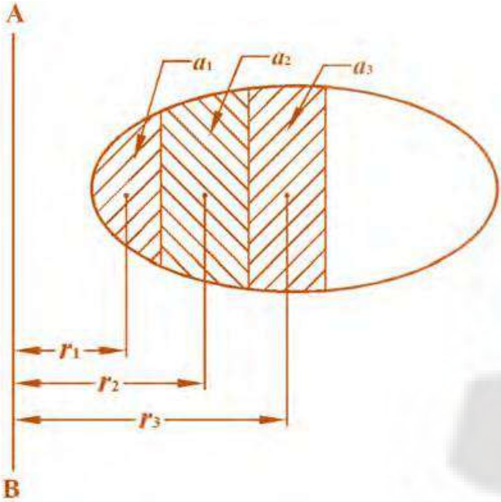


Fig.5.2 Moment of inertia of a plane figure

Consider a plane figure of area A whose moment of inertia is required to be found out. Divide the plane area into number of small elemental strips as shown in fig.5.2.

Let a_1, a_2, a_3, etc. be the areas of the elemental strips and r_1, r_2, r_3,, etc. be the distance of their centroids from a fixed line AB.

First moment of area of the first strip about AB = $a_1 r_1$

The second moment of area of the first strip about AB

$$= a_1 . r_1 . r_1 = a_1 . r_1^2$$

∴ The second moment of area of the plane figure about AB

$$= a_1 r_1^2 + a_2 r_2^2 + \cdots = \Sigma a . r^2$$

This second moment of area is known as moment of inertia.

2

5.6 Parallel axis theorem

It states, *if the moment of inertia of a plane area about an axis passing through its centroid is denoted by I_G then the moment of inertia of the area about any other axis AB which is parallel to the first and at a distance h from the centroidal is given by,*

$$I_{AB} = I_G + Ah^2$$

Where, I_{AB} = Moment of inertia of the area about an axis AB.

I_G = Moment of inertia of the area about its centroid

A = Area of the section

h = Distance between centroid of the section and axis AB.

Proof

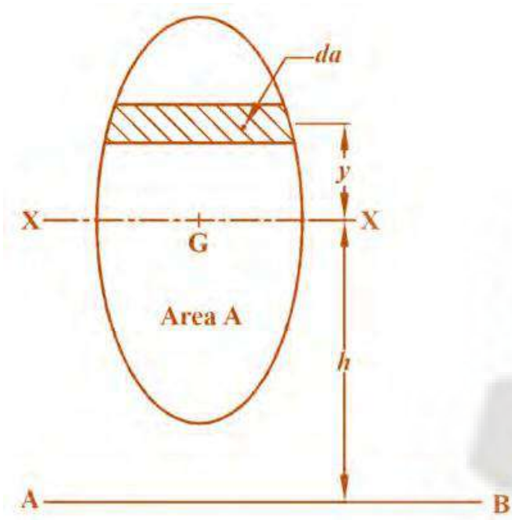


Fig.5.3 Parallel axis theorem

Consider an elemental strip in a plane whose moment of inertia is required to be found out about an axis AB as shown in the fig.5.3

Let, ða = Area of the strip

y = Distance of C.G of strip from C.G of the section

h = Distance of axis AB from the C.G of section.

We know that, the moment of inertia of the elemental strip about an axis passing through the C.G of the section,

$$I = ða.\, y^2$$

Moment of inertia of the whole section about an axis passing through the C.G of the section,

$$I_G = \Sigma ða\, y^2$$

The moment of inertia of the section about the axis AB,

$$I_{AB} = \Sigma ða(h + y)^2 = \Sigma ða(h^2 + y^2 + 2hy)$$

$$= h^2\Sigma ða + y^2\Sigma ða + 2hy\Sigma ða$$

$$= Ah^2 + I_G + 0$$

$\Sigma ða.\, y = Ay = 0$ *(‡ First moment of area about centroidal axis = 0)*

$$\therefore I_{AB} = I_G + Ah^2$$

5.7 Perpendicular axis theorem

It states, *if* I_{xx} *and* I_{yy} *be the moments of inertia of plane section about two perpendicular axes meeting at O, the moment of inertia* I_{ss} *about the axis Z–Z, perpendicular to the plane and passing through the intersection of X–X and Y–Y axes is given by,*

$$\therefore I_{77} = I_{zz} + I_{yy}$$

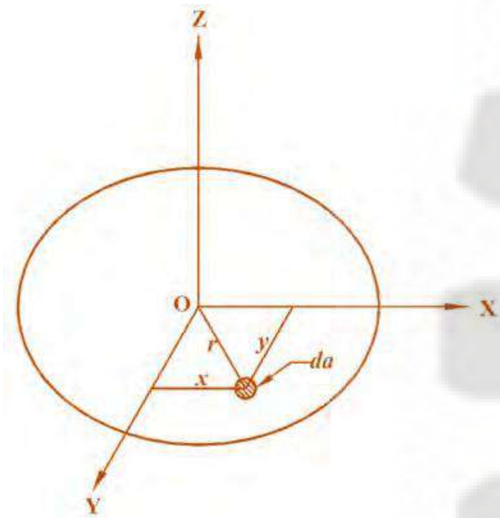


Fig.5.4 Perpendicular axis theorem

Proof

Consider three mutually perpendicular axes OX, OY and OZ. Consider
a small lamina of area *da* having co–ordinates as *x* and *y* along OX and OY. Let r be the distance of the lamina form Z–Z axis.

From the geometry of the figure, $r^2 = x^2 + y^2$

The moment of inertia of the lamina about X–X axis is given by,

$$I_{xx} = da.\, y^2$$

Similarly, $I_{yy} = da.\, x^2$

$$I_{ss} = da.\, r^2 = da\,(x^2 + y^2)$$

$$= da.\, x^2 + da.\, y^2 = I_{xx} + I_{yy}$$

$$\therefore I_{77} = I_{zz} + I_{yy}$$

5.8 Derivation of moment of inertia of some sections

1) Rectangular section

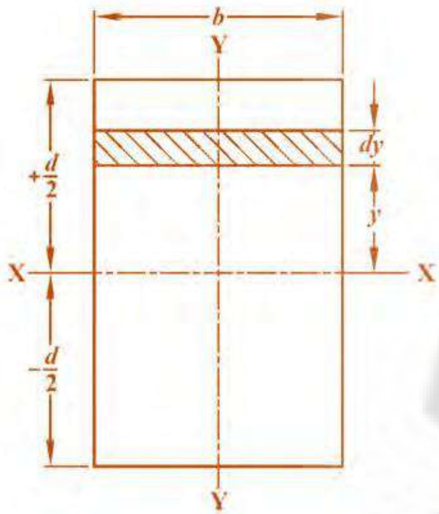


Fig.5.5 M.I of rectangular section

Consider a rectangular section of width b and depth d as shown in the fig.5.5. Now consider an elemental strip of thickness dy parallel to X–X axis and at a distance y from X–X axis.

Area of the strip = b. dy

M I of the strip about X–X axis = Area × $(\text{Distance})^2$

$= b. dy. y^2 = by^2dy$

M. I of the whole section about X–X axis,

$$I_{xx} = \int_{-\frac{d}{2}}^{+\frac{d}{2}} by^2 dy = b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} = b\left[\frac{\frac{d^3}{8}}{3} + \frac{\frac{d^3}{8}}{3}\right]$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right] = b\left[\frac{2d^3}{24}\right]$$

$$I_{zz} = \frac{bd^3}{12}\ ;$$

Similarly,

$$I_{yy} = \frac{db^3}{12}$$

2) Circular section

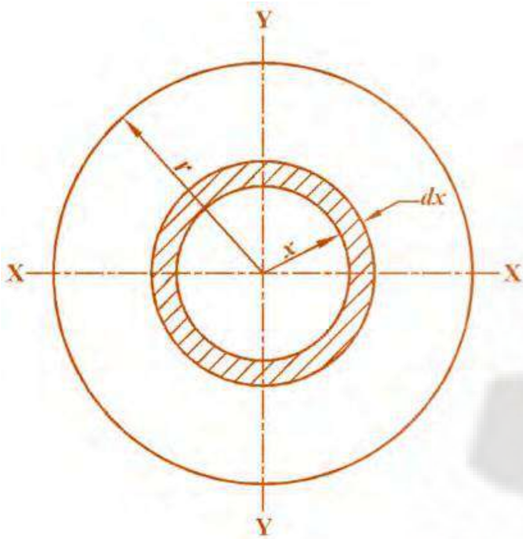


Fig.5.6 M.I of circular section

Consider a circle of radius r with centre O and X–X and Y–Y be the two axes of reference passing through O.

Now consider an elementary ring of radius *x* and thickness dx.

∴ The area of the ring, da = $2\pi x \cdot dx$

Moment of inertia of the ring about Z–Z axis

$$= \text{Area} \times (\text{Distance})^2 = 2\pi x \cdot dx \cdot x^2 = 2\pi x^3 dx$$

The moment of inertia of whole section about Z–Z axis

$$I_{ss} = \int_0^r 2\pi x^3\, dx = \left[\frac{2\pi x^4}{4}\right]_0^r = \frac{2\pi r^4}{4} = \frac{\pi r^4}{2}$$

Substituting, $r = \frac{d}{2}$,

$$I_{ss} = \frac{\pi (d/2)^4}{2} = \frac{\pi d^4}{32}$$

From the geometry of the section, $I_{xx} = I_{yy}$. According to perpendicular axis theorem,

$$I_{ss} = I_{xx} + I_{yy} = 2I_{xx} \text{ or } 2I_{yy}$$

$$I_{zz} = I_{yy} = \frac{I_{ss}}{2} = \frac{\pi (d^4/32)}{2} = \frac{\pi d^4}{64}$$

3) Triangular section

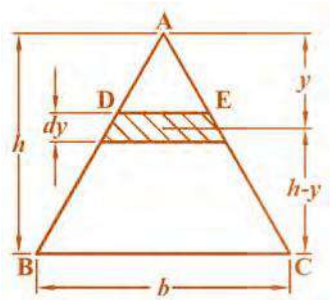


Fig.5.7 M.I of triangular section

Consider a triangular section ABC of base b and height h.

Consider an elemental strip DE of thickness *dy* at a distance of y from the vertex A as shown in the fig.5.7.

From the figure, the triangle ADE and ABC are similar.

$$\therefore \frac{DE}{BC} = \frac{y}{h}$$

$$DE = BC \cdot \frac{y}{h} = \frac{by}{h}$$

Area of the strip, $da = \frac{by}{h} dy$

Moment of inertia of the strip about the base BC

$$= \text{Area} \times (\text{Distance})^2 = \frac{by}{h} dy\,(h - y)^2$$

Moment of inertia of the whole section about the base BC,

$$I_{BC} = \int_0^h \frac{by}{h} (h - y)^2 dy$$

$$I_{BC} = \frac{b}{h} \int_0^h y(h^2 + y^2 - 2hy) dy$$

$$I_{BC} = \frac{b}{h} \int_0^h (yh^2 + y^3 - 2hy^2) dy$$

$$= \frac{b}{h} \left[\frac{y^2h^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3} \right]_0^h$$

$$= \frac{b}{h}\left[\frac{h^4}{2} + \frac{h^4}{4} - \frac{2h^4}{3}\right]$$

$$= \frac{b}{h}\left[\frac{6h^4 + 3h^4 - 8h^4}{12}\right]$$

$$= \frac{bh^4}{12h} = \frac{bh^3}{12}$$

$$\therefore I_{BC} = \frac{bh^3}{12}$$

The moment of inertia of a triangular section about the axis passing through it centre of gravity.

In a triangular section, the distance of C.G from the base is given by,

$$h_1 = \frac{h}{3}$$

According to the parallel axis theorem,

$$I_{BC} = I_G + a h_1^2$$

$$I_G = I_{BC} - a h_1^2$$

$$= \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{3bh^3 - 2bh^3}{36} = \frac{bh^3}{36}$$

$$\therefore I_G = \frac{bh^3}{36}$$

5.9 Polar moment of inertia

The moment of inertia of a plane area with respect to the centroidal axis perpendicular to the plane area is called *polar moment of inertia*.

Mathematically, I_P or $J = I_{xx} + I_{yy}$

For a circular section, $J = \frac{\pi d^4}{32}$

5.10 Radius of gyration

Radius of gyration may be defined as *the distance at which the whole area of the plane figure is assumed to be concentrated with respect to a reference axis.*

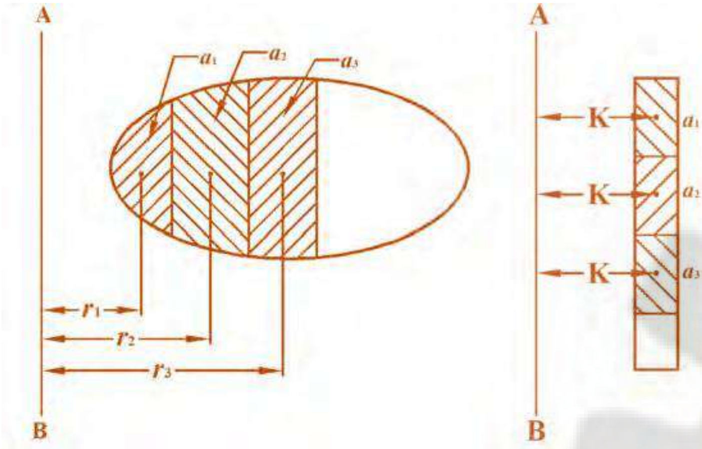


Fig.5.8 Radius of gyration

Consider a plane figure of area A. Divide the whole area into number of vertical strips as shown in the fig.5.8. Let a_1, a_2, a_3, etc. be the area of the strips and r_1, r_2, r_3, ..., etc. be the distance of these areas from a given axis AB.

The moment of inertia of the area about the reference axis AB, $I_{AB} = \Sigma ar^2$

Let us assume that the vertical strips be arranged at the same distance K from the axis AB so that the moment of inertia about the axis AB remains unchanged. Now the moment of inertia of the plane figure about the axis AB,

$$\therefore I_{AB} = a_1K^2 + a_2K^2 + a_3K^2 + \cdots = K^2\Sigma a = AK^2$$

$$I_{AB} = AK^2 \text{ (or) } K = \sqrt{\frac{I_{AB}}{A}}$$

Where, K is *radius of gyration* of the plane figure about the axis AB.

5.11 Section modulus

The section modulus or modulus of section is the ratio between the moment of inertia of the figure about its centroidal axis and the distance of extreme surface from the centroidal axis. It is usually denoted by Z.

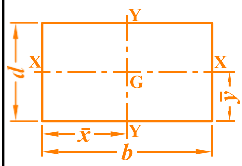
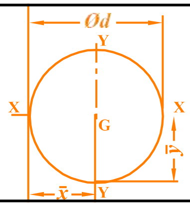
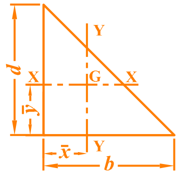
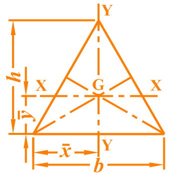
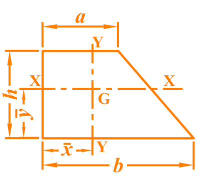
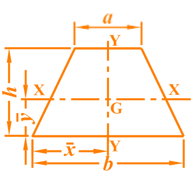
$$\therefore Z = \frac{\text{Moment of inertia about centroidal axis}}{\text{Distance of extreme surface from centroidal axis}}$$

Section modulus of rectangle, Z $= \frac{I_G}{d/2} = \frac{bd^3}{12} \times \frac{2}{d} = \frac{bd^2}{6}$

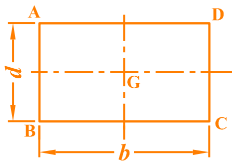
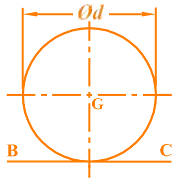
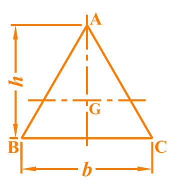
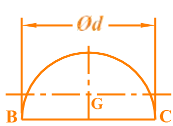
Section modulus of circle, Z $= \frac{I_G}{d/2} = \frac{\pi d^4}{64} \times \frac{2}{d} = \frac{\pi d^3}{32}$

POINTS TO REMEMBER

1) Position of centroid of plane geometrical figures

Shape	Figure	Area	$\bar{X}$	$\bar{Y}$
Rectangle		bd	$\frac{b}{2}$	$\frac{d}{2}$
Circle		$\frac{vd^2}{4}$	$\frac{d}{2}$	$\frac{d}{2}$
Triangle		$\frac{bh}{2}$	$\frac{b}{3}$	$\frac{h}{3}$
Triangle		$\frac{bh}{2}$	Intersection of medians	$\frac{h}{3}$
Trapezium		$\frac{(a+b)h}{2}$	$\frac{(a^2+b^2+ab)}{3(a+b)}$	$\frac{(2a+b)h}{3(a+b)}$
Trapezium		$\frac{(a+b)h}{2}$	$\frac{b}{2}$	$\frac{(2a+b)h}{3(a+b)}$

2) Moment of inertia of plane geometrical figures

Shape	Figure	M.I about centroidal axis (I_G)	M.I about base (I_{BC})
Rectangle		$I_G = \frac{bd^3}{12}$	$I_{BC} = \frac{bd^3}{3}$
Circle		$I_G = \frac{vd^4}{64}$	$J = \frac{vd^4}{32}$
Triangle		$I_G = \frac{bh^3}{36}$	$I_{BC} = \frac{bh^3}{12}$
Semi circle		$I_G = \frac{vd^4}{24} - \frac{d^4}{18v}$	$I_{BC} = \frac{vd^4}{128}$

3) $\bar{X} = \frac{a_1 z_1 + a_2 z_2 + a_3 z_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$ (mm)

4) $\bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$ (mm)

5) Parallel axis theorem, $I_{AB} = I_G + ah^2$ (mm^4)

6) Perpendicular axis theorem, $I_{77} = I_{zz} + I_{yy}$ (mm^4)

7) Radius of gyration, $K = \sqrt{\frac{I}{A}}$ (mm)

SOLVED PROBLEMS

DETERMINATION OF CENTROID

Example : 5.1

Determine the centroid of an angle section 100mm × 80mm × 20mm thick with its longer arm being placed vertical.

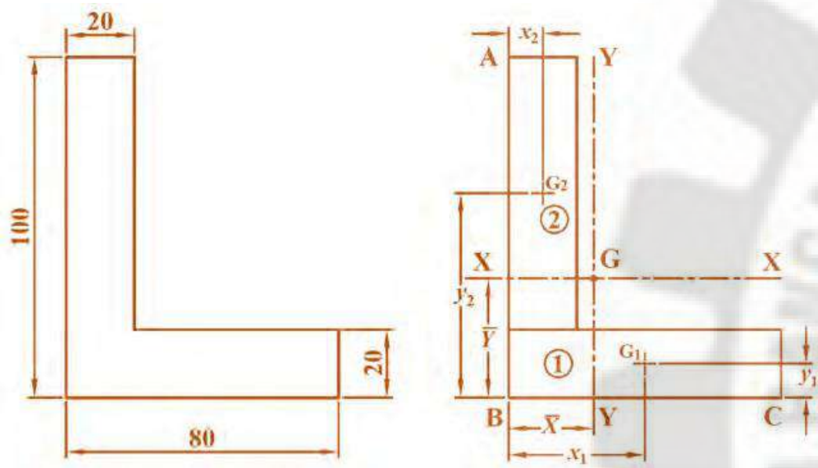


Fig.P5.1 Centroid of 'L' section [Example 5.1]

Solution :

Split the section into two rectangles as shown. Let, AB and BC be the reference axes

Let $\bar{X}$ *and* $\bar{Y}$ *be the distance of C.G from AB and BC respectively.*

$a_1 = 80 \times 20 = 1600\ mm^2;\ x_1 = \frac{80}{2} = 40\ mm;\quad y_1 = \frac{20}{2} = 10\ mm$

$a_2 = 20 \times 80 = 1600\ mm^2;\ x_2 = \frac{20}{2} = 10\ mm;\ y_2 = 20 + \frac{80}{2} = 60\ mm$

$$\bar{X} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{(1600 \times 40) + (1600 \times 10)}{1600 + 1600} = \frac{80000}{3200} = \boxed{25\ \mathbf{mm}}$$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(1600 \times 10) + (1600 \times 60)}{1600 + 1600} = \frac{112000}{3200} = \boxed{35\ \mathbf{mm}}$$

Result : The coordinate of centroid from reference axes

$\bar{X} = 25$ **mm** and $\bar{Y} = 35$ **mm**

Example : 5.2

Find the centroid of the section shown in the fig.P5.2

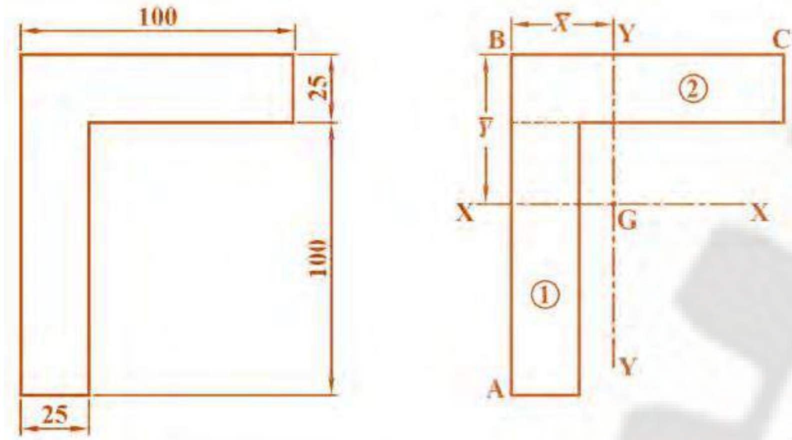


Fig.P5.2 Centroid of 'L' section [Example 5.2]

Solution :

$a_1 = 25 \times 100 = 2500\ mm^2;\ a_2 = 100 \times 25 = 2500\ mm^2$

$x_1 = \frac{25}{2} = 12.5\ mm;\ y_1 = 25 + \frac{100}{2} = 75\ mm$

$x_2 = \frac{100}{2} = 50\ mm;\ y_2 = \frac{25}{2} = 12.5\ mm$

$\bar{X} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{(2500 \times 12.5) + (2500 \times 50)}{2500 + 2500} = \frac{156250}{5000} =$ **31. 25 mm**

$\bar{Y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(2500 \times 75) + (2500 \times 12.5)}{2500 + 2500} = \frac{218750}{5000} =$ **43. 75 mm**

Result : $\bar{X}$ = 31. 25 **mm** and $\bar{Y}$ = 43. 75 **mm** from reference axes

Example : 5.3

(Apr.14)

Find the centroid of a T–section with flange 100mm × 30mm and web 120mm × 30mm.

Solution :

This section is symmetrical about Y–Y axis. So the C.G will lie on this axis

$\therefore \bar{X} = \frac{100}{2} =$ **50 mm**

$a_1 = 100 \times 30 = 3000\ mm^2;\ a_2 = 30 \times 120 = 3600\ mm^2$

$y_1 = \frac{30}{2} = 15\ mm;\ y_2 = 30 + \frac{120}{2} = 90\ mm$

$\bar{Y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(3000 \times 15) + (3600 \times 90)}{3000 + 3600} = \frac{369000}{6600} =$ **55. 91 mm**

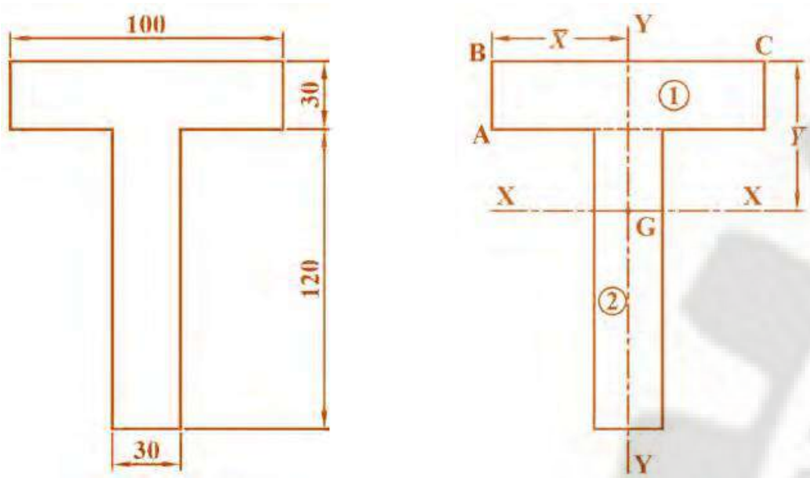


Fig.P5.3 Centroid of 'T' section [Example 5.3]

Result : $\bar{X} = 50$ **mm** and $\bar{Y} = 55.91$ **mm** from reference axes

Example : 5.4

(Apr.04, Oct.12)

Find the centroid of an inverted T–section with flange 150mmx20mm and web 100mm × 25mm.

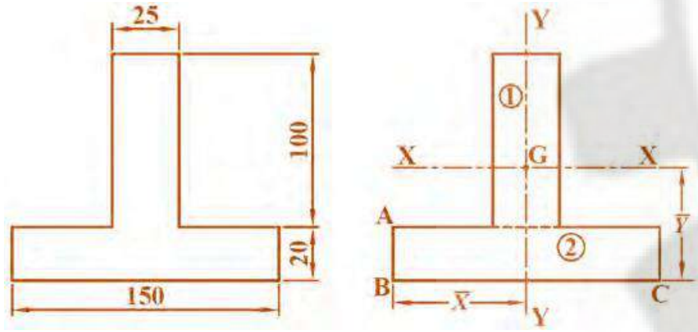


Fig.P5.4 Centroid of inverted 'T' section [Example 5.4]

Solution :

This section is symmetrical about Y–Y axis. So the C.G will lie on this axis

$$\therefore \bar{X} = \frac{150}{2} = \boxed{75 \text{ mm}}$$

$$a_1 = 25 \times 100 = 2500 \text{ mm}^2;\ a_2 = 150 \times 20 = 3000 \text{ mm}^2$$

$$y_1 = 20 + \frac{100}{2} = 70 \text{ mm};\ y_2 = \frac{20}{2} = 10 \text{ mm}$$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(2500 \times 70) + (3000 \times 10)}{2500 + 3000} = \frac{205000}{5500} = \boxed{37.273 \text{ mm}}$$

Result : $\bar{X}$ = 75 **mm** and $\bar{Y}$ = 37.273 **mm** from reference axes

Example : 5.5

A channel section of size 100mm × 50mm overall. The base as well as the flanges of the channel are 15mm thick. Determine the centroid for the section.

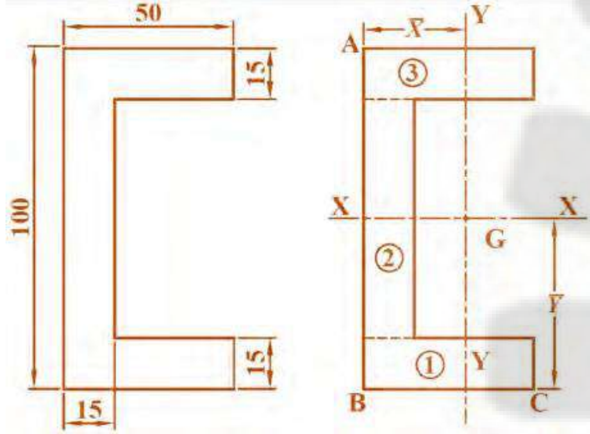


Fig.P5.5 Centroid of channel section [Example 5.5]

Solution :

This section is symmetrical about X–X axis. So the C.G will lie on this axis

$$\therefore \bar{Y} = \frac{100}{2} = \boxed{50 \text{ mm}}$$

a_1=50×15=750 mm^2; a_2=70×15=1050 mm^2; a_3=50×15=750 mm^2

$x_1 = \frac{50}{2}$ = 25 mm; $x_2 = \frac{15}{2}$ = 7.5 mm; $x_3 = \frac{50}{2}$ = 25 mm

$$\bar{X} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = \frac{(750 \times 25) + (1050 \times 7.5) + (750 \times 25)}{750 + 1050 + 750}$$

$$= \frac{45375}{2550} = \boxed{17.794 \text{ mm}}$$

Result : $\bar{X}$ = 17.794 **mm** and $\bar{Y}$ = 50 **mm** from reference axes

Example : 5.6

(Oct.14)

Find the centroid of an I–section having top flange 150mm × 25mm, web 160mm × 25mm and bottom flange 200mm × 25mm.

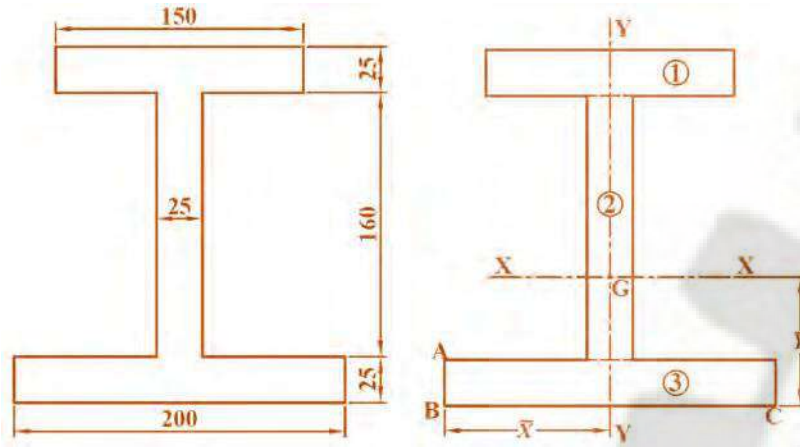


Fig.P5.6 Centroid of 'I' section [Example 5.6]

Solution :

This section is symmetrical about Y–Y axis. So the C.G will lie on this axis $\therefore \bar{X} = \frac{200}{2} =$ **100 mm**

$a_1 = 150 \times 25 = 3750\ mm^2$; $y_1 = 25 + 160 + \frac{25}{2} = 197.5\ mm$

$a_2 = 25 \times 160 = 4000\ mm^2$; $y_2 = 25 + \frac{160}{2} = 105\ mm$

$a_3 = 200 \times 25 = 5000\ mm^2$; $y_3 = \frac{25}{2} = 12.5\ mm$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{(3750 \times 197.5) + (4000 \times 105) + (5000 \times 12.5)}{3750 + 4000 + 5000} = \frac{1223125}{12750} = 95.931\ mm$$

Result : $\bar{X}$ = 100 **mm** and $\bar{Y}$ = 95.931 mm from reference axes

DETERMINATION MOMENT OF INERTIA

Example : 5.7 *(Oct.01)*

Determine the polar moment of inertia of rectangle 100mm ×150mm.

Solution :

Moment of inertia of rectangular section about X–X axis,

$$I_{xx} = \frac{bd^3}{12} = \frac{100 \times 150^3}{12} = 28125000\ mm^4$$

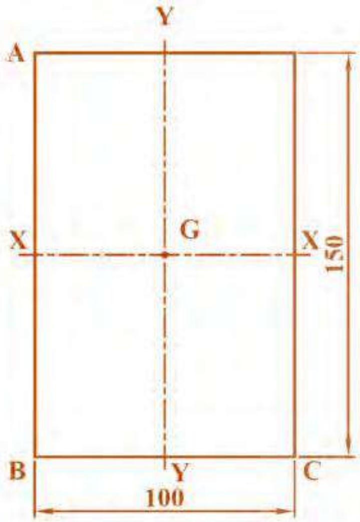


Fig.P5.7 M.I of rectangular section [Example 5.7]

Moment of inertia of rectangular section about Y–Y axis,

$$I_{yy} = \frac{db^3}{12} = \frac{150 \times 100^3}{12} = 12500000\ mm^4$$

Polar moment of inertia,

$$I_{ss} = I_{xx} + I_{yy} = 28125000 + 12500000 = \boxed{40625000\ mm^4}$$

Result : The polar moment of inertia, I_{77} = 40625000 mm^4

Example : 5.8

(Apr.01)

Determine the polar moment of inertia of a circle of diameter 100mm.

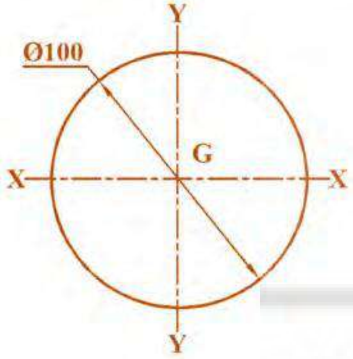


Fig.P5.8 M.I of circular section [Example 5.8]

Solution :

Diameter of the circle, d = 100 mm

Moment of inertia of circular section about X–X or Y–Y axis,

$$I_{xx} = I_{yy} = \frac{\pi d^4}{64} = \frac{\pi \times 100^4}{64} = 4908738.521 \text{ mm}^4$$

Polar moment of inertia,

$$I_{ss} = I_{xx} + I_{yy} = 4908738.521 + 4908738.521 = \boxed{9817477.042 \text{ mm}^4}$$

Result : The polar moment of inertia, I_{zz} = 9817477.042 mm^4

Example : 5.9

(Apr.03, Oct.16)

An angle section is of 100 mm wide and 120 mm deep overall. Both the flanges of the angle are 10 mm thick. Determine the moment of inertia about the centroidal axes X-X and Y-Y. Also find its radius of gyration about its centroidal axes.

Solution :

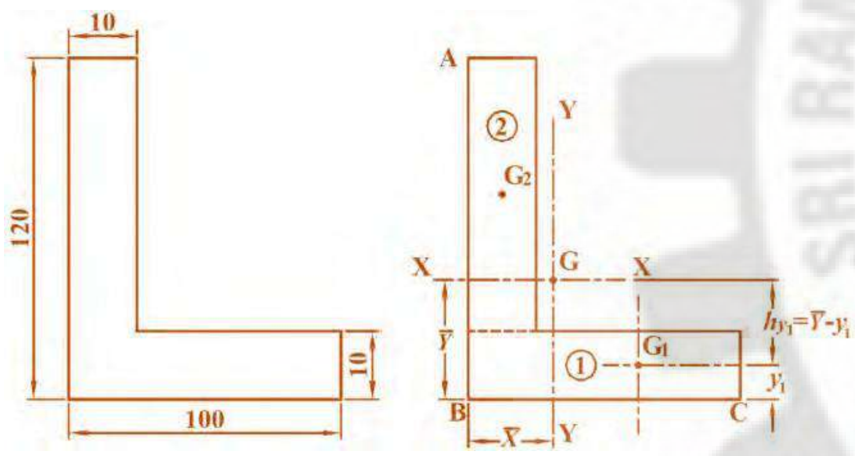


Fig.P5.9 M.I of 'L' section [Example 5.9]

Split the section into two rectangles as shown.

$$a_1 = 100 \times 10 = 1000 \text{ mm}^2; \; x_1 = \frac{100}{2} = 50 \text{ mm}; \; y_1 = \frac{10}{2} = 5 \text{ mm}$$

$$a_2 = 10 \times 110 = 1100 \text{ mm}^2; \; x_2 = \frac{10}{2} = 5 \text{ mm}; \; y_2 = 10 + \frac{110}{2} = 65 \text{ mm}$$

$$\bar{X} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{(1000 \times 50) + (1100 \times 5)}{1000 + 1100} = \frac{55500}{2100} = 26.43 \text{ mm}$$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(1000 \times 5) + (1100 \times 65)}{1000 + 1100} = \frac{76500}{2100} = 36.43 \text{ mm}$$

Calculation for I_{zz}

Distance of C.G of section (1) from X–X axis,

$$h_{y1} = \bar{Y} - y_1 = 36.43 - 5 = 31.43 \text{ mm}$$

Distance of C.G of section (2) from X–X axis,

$$h_{y2} = \bar{Y} - y_2 = 36.43 - 65 = -28.57 \text{ mm}$$

Moment of inertia of section (1) about an axis parallel to X–X and passing through its C.G (G_1),

$$I_{Gx1} = \frac{b_1 d_1^3}{12} = \frac{100 \times 10^3}{12} = 8333.333 \text{ mm}^4$$

Moment of inertia of section (2) about an axis parallel to X–X and passing through its C.G (G_2),

$$I_{Gx2} = \frac{b_2 d_2^3}{12} = \frac{10 \times 110^3}{12} = 1109166.667 \text{ mm}^4$$

According to parallel axis theorem,

the moment of inertia of section (1) about X–X axis,

$$I_{xx1} = I_{Gx1} + a_1 h_{y1}^2 = 8333.333 + [1000 \times 31.43^2] = 996178.233 \text{ mm}^4$$

Similarly,

$$I_{xx2} = I_{Gx2} + a_2 h_{y2}^2 = 1109166.667 + [1100 \times (-28.57)^2] = 2007036.057 \text{ mm}^4$$

Moment of inertia of the whole section about X–X axis,

$$I_{xx} = I_{xx1} + I_{xx2} = 996178.233 + 2007036.057 = 3003214.29$$

$$= \boxed{3.0032 \times 10^6 \text{ mm}^4}$$

Calculation for I_{yy}

Distance of C.G of section (1) from Y–Y axis,

$$h_{x1} = \bar{X} - x_1 = 26.43 - 50 = -23.57 \text{ mm}$$

Distance of C.G of section (2) from Y–Y axis,

$$h_{x2} = \bar{X} - x_2 = 26.43 - 5 = 21.43 \text{ mm}$$

Moment of inertia of section (1) about an axis parallel to Y–Y and passing through its C.G(G_1),

$$I_{Gy1} = \frac{d_1 b_1^3}{12} = \frac{10 \times 100^3}{12} = 833333.333 \text{ mm}^4$$

Moment of inertia of section (2) about an axis parallel to Y–Y and passing through its C.G(G_2),

$$I_{Gy2} = \frac{d_2 b_2^3}{12} = \frac{110 \times 10^3}{12} = 9166.667 \text{ mm}^4$$

According to parallel axis theorem,

the moment of inertia of section (1) about Y–Y axis,

$I_{yy1} = I_{Gy1} + a_1 h_{x1}^2 = 833333.333 + [1000 \times (-23.57)^2] = 1388878.233\ mm^4$

$I_{yy2} = I_{Gy2} + a_2 h_{x2}^2 = 9166.667 + [1100 \times 21.43^2] = 514336.057\ mm^4$

Moment of inertia of the whole section about Y–Y axis,

$I_{yy} = I_{yy1} + I_{yy2} = 1388878.233 + 514336.057 = 1903214.29$

$= \boxed{1.9032 \times 10^6\ mm^4}$

Calculation for K_{zz}

Radius of gyration about centroidal axis X-X,

$K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{3003214.29}{2100}} = \boxed{37.817\ mm}$

Calculation for K_{yy}

Radius of gyration about centroidal axis Y-Y,

$K_{yy} = \sqrt{\frac{I_{yy}}{\Sigma a}} = \sqrt{\frac{1903214.29}{2100}} = \boxed{30.105\ mm}$

Result : 1) The moment of inertia about centroidal axes, $I_{zz} = 2.088 \times 10^6\ mm^4$; $I_{yy} = 1.2974 \times 10^6\ mm^4$ 2) The radius of gyration about centroidal axes, $K_{zz} = 37.817\ mm$; $K_{yy} = 30.105\ mm$

Example : 5.10

(Oct.03, Oct.04, Apr.13, Apr.18)

Find the values of I_{zz} and I_{yy} of a T–section 120mm wide and 120mm deep overall. Both the web and flange are 10mm thick. Also calculate K_{zz} and K_{yy}.

Solution :

This section is symmetrical about Y–Y axis. So the C.G will lie on this axis. $\therefore \bar{X} = \frac{120}{2} = 60\ mm$

$a_1 = 120 \times 10 = 100\ mm^2$; $a_2 = 10 \times 110 = 1100\ mm^2$

$y_1 = \frac{10}{2} = 5\ mm$; $y_2 = 10 + \frac{110}{2} = 65\ mm$

$\bar{Y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(1200 \times 5) + (1100 \times 65)}{1200 + 1100} = \frac{77500}{2300} = 33.696\ mm$

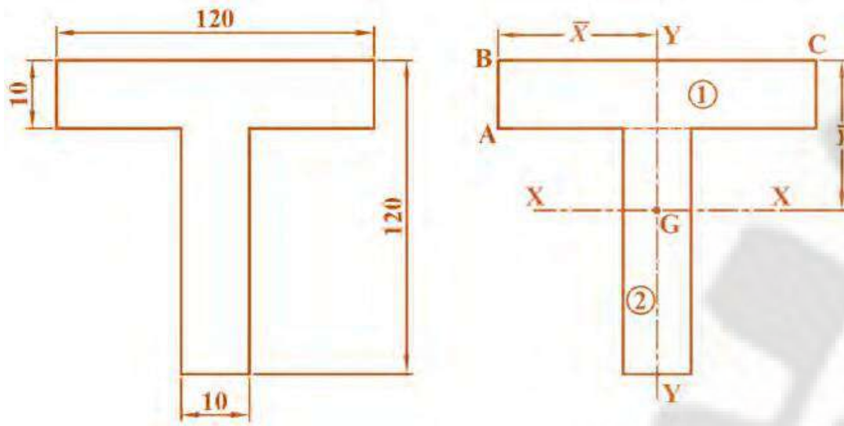


Fig.P5.10 M.I of 'T' section [Example 5.10]

Calculation for I_{zz}

$$h_{y1} = \bar{Y} - y_1 = 33.696 - 5 = 28.696 \text{ mm}$$

$$h_{y2} = \bar{Y} - y_2 = 33.696 - 65 = -31.304 \text{ mm}$$

$$I_{Gx1} = \frac{b_1 d_1^3}{12} = \frac{120 \times 10^3}{12} = 10000 \text{ mm}^4$$

$$I_{Gx2} = \frac{b_2 d_2^3}{12} = \frac{10 \times 110^3}{12} = 1109166.667 \text{ mm}^4$$

$$I_{xx1} = I_{Gx1} + a_1 h^2_{y1} = 10000 + [1200 \times (28.696)^2] = 998152.5 \text{ mm}^4$$

$$I_{xx2} = I_{Gx2} + a_2 h^2_{y2}$$

$$= 1109166.667 + [1100 \times (-31.304)^2] = 2187101.125 \text{ mm}^4$$

$$I_{xx} = I_{xx1} + I_{xx2} = 998152.5 + 2187101.125 = \boxed{3.185 \times 10^6 \text{ mm}^4}$$

Calculation for I_{yy}

$$h_{x1} = \bar{X} - x_1 = 60 - 60 = 0$$

$$h_{x2} = \bar{X} - x_2 = 60 - 60 = 0$$

$$I_{Gy1} = \frac{d_1 b_1^3}{12} = \frac{10 \times 120^3}{12} = 144000 \text{ mm}^4$$

$$I_{Gy2} = \frac{d_2 b_2^3}{12} = \frac{110 \times 10^3}{12} = 9166.667 \text{ mm}^4$$

$$I_{yy1} = I_{Gy1} + a_1 h^2_{x1} = 144000 + 0 = 1440000 \text{ mm}^4$$

$$I_{yy2} = I_{Gy2} + a_2 h^2_{x2} = 9166.667 + 0 = 9166.667 \text{ mm}^4$$

$$I_{yy} = I_{yy1} + I_{yy2} = 1440000 + 9166.667 = \boxed{1.449 \times 10^6\ mm^4}$$

Calculation for radius of gyration

$$K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{3.185 \times 10^6}{2300}} = \boxed{37.213\ mm}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{\Sigma a}} = \sqrt{\frac{1.449 \times 10^6}{2300}} = \boxed{25.1\ mm}$$

Result : 1) $I_{zz} = 3.185 \times 10^6 mm^4$	2) $I_{yy} = 1.449 \times 10^6\ mm^4$
3) $K_{zz} = 37.213\ mm$	4) $K_{yy} = 25.1\ mm$

Example : 5.11

(Apr.90)

Calculate I_{zz} and I_{yy} for the section shown in the fig.P5.11. Also find K_{zz} and K_{yy}.

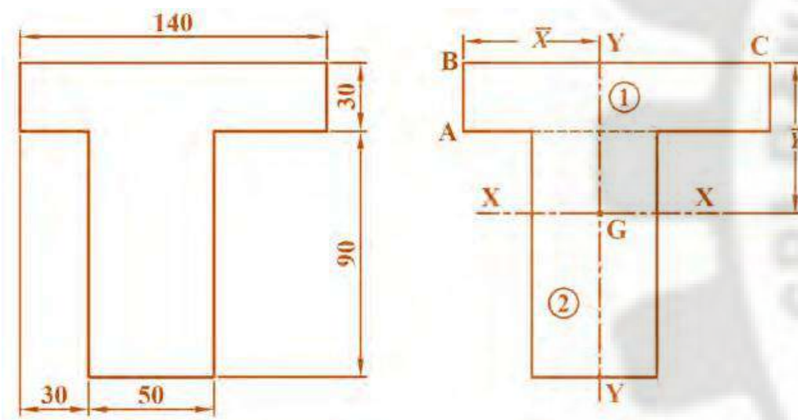


Fig.P5.11 M.I of 'T' section [Example 5.11]

Solution :

$a_1 = 140 \times 30 = 4200\ mm^2$; $x_1 = \frac{140}{2} = 70\ mm$; $y_1 = \frac{30}{2} = 15\ mm$

$a_2 = 50 \times 90 = 4500\ mm^2$; $x_2 = 30 + \frac{50}{2} = 55\ mm$; $y_2 = 30 + \frac{90}{2} = 75\ mm$

$$\bar{X} = \frac{a_1x_1 + a_2x_2}{a_1 + a_2} = \frac{(4200 \times 70) + (4500 \times 55)}{4200 + 4500} = \frac{5415100}{8700} = 64.241\ mm$$

$$\bar{Y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2} = \frac{(4200 \times 15) + (4500 \times 75)}{4200 + 4500} = \frac{400500}{8700} = 46.034\ mm$$

Calculation for I_{zz}

$h_{y1} = \bar{Y} - y_1 = 46.034 - 15 = 31.034$ mm

$h_{y2} = \bar{Y} - y_2 = 46.034 - 75 = -28.966$ mm

$$I_{Gx1} = \frac{b_1 d_1^3}{12} = \frac{140 \times 30^3}{12} = 315000 \text{ mm}^4$$

$$I_{Gx2} = \frac{b_2 d_2^3}{12} = \frac{50 \times 90^3}{12} = 3.0375 \times 10^6 \text{ mm}^4$$

$I_{xx1} = I_{Gx1} + a_1 h^2_{y1}$

$= 315000 + [4200 \times (31.034)^2] = 4360058.455 \text{ mm}^4$

$I_{xx2} = I_{Gx2} + a_2 h^2_{y2}$

$= 3.0375 \times 10^6 + [4500 \times (-28.966)^2] = 6813131.202 \text{ mm}^4$

$I_{xx} = I_{xx1} + I_{xx2} = 4360058.455 + 6813131.202 =$ **$11.173 \times 10^6 \text{ mm}^4$**

Calculation for I_{yy}

$h_{x1} = \bar{X} - x_1 = 62.241 - 70 = -7.759$ mm

$h_{x2} = \bar{X} - x_2 = 62.241 - 55 = 7.241$ mm

$$I_{Gy1} = \frac{d_1 b_1^3}{12} = \frac{30 \times 140^3}{12} = 6.86 \times 10^6 \text{ mm}^4$$

$$I_{Gy2} = \frac{d_2 b_2^3}{12} = \frac{90 \times 50^3}{12} = 937500 \text{ mm}^4$$

$I_{yy1} = I_{Gy1} + a_1 h^2_{x1} = 6.86 \times 10^6 + [4200 \times (-7.759)^2] = 7112848.74 \text{ mm}^4$

$I_{yy2} = I_{Gy2} + a_2 h^2_{x2} = 937500 + [4500 \times (7.241)^2] = 1173444.365 \text{ mm}^4$

$I_{yy} = I_{yy1} + I_{yy2} = 7112848.74 + 1173444.365 =$ **$8.2863 \times 10^6 \text{ mm}^4$**

Calculation for radius of gyration

$$K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{11.173 \times 10^6}{870}} = 35.836 \text{ mm}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{\Sigma a}} = \sqrt{\frac{8.2863 \times 10^6}{870}} = 30.862 \text{ mm}$$

Result :
1) $I_{zz} = 11.173 \times 10^6 \text{ mm}^4$ 2) $I_{yy} = 8.2863 \times 10^6 \text{ mm}^4$
3) $K_{zz} = 35.836$ **mm** 4) $K_{yy} = 30.862$ **mm**

Example : 5.12 *(Apr.05, Oct.12)*

A channel section is of size 300mm×100mm overall. The base as well as the flanges of the channel are 10mm thick. Determine the values of I_{zz} and I_{yy}. A.lso find K_{zz} and K_{yy}.

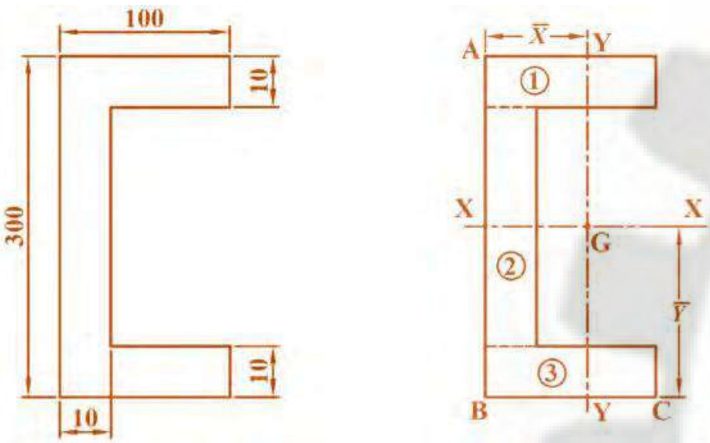


Fig.P5.12 M.I of channel section [Example 5.12]

Solution :

This section is symmetrical about X–X axis. So the C.G will lie on this axis $\therefore \bar{Y} = \frac{300}{2} = 150$ mm

$$a_1 = a_3 = 100 \times 10 = 1000 \text{ mm}^2;\ a_2 = 10 \times 280 = 2800 \text{ mm}^2$$

$$x_1 = x_3 = \frac{100}{2} = 50 \text{ mm};\ x_2 = \frac{10}{2} = 5 \text{ mm}$$

$$\bar{X} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{(1000 \times 50) + (2800 \times 5) + (1000 \times 50)}{1000 + 2800 + 1000} = \frac{114000}{4800} = 23.75 \text{ mm}$$

Calculation for I_{zz}

$$h_{y1} = \bar{Y} - y_1 = 150 - 5 = 145 \text{ mm}$$

$$h_{y2} = \bar{Y} - y_2 = 150 - \left(10 + \frac{280}{2}\right) = 0 \text{ mm}$$

$$h_{y3} = \bar{Y} - y_3 = 150 - \left(10 + 280 + \frac{10}{2}\right) = -145 \text{ mm}$$

$$I_{Gx1} = I_{Gx3} = \frac{b_1 d_1^3}{12} = \frac{100 \times 10^3}{12} = 8333.333 \text{ mm}^4$$

$$I_{Gx2} = \frac{b_2 d_2^3}{12} = \frac{10 \times 280^3}{12} = 18.2933 \times 10^6 \text{ mm}^4$$

$$I_{xx1} = I_{Gx1} + a_1 h_{y1}^2 = 83333.333 + [1000 \times (145)^2] = 21.0333 \times 10^6 \text{ mm}^4$$

$$I_{xx2} = I_{Gx2} + a_2 h_{y2}^2 = 18.2933 \times 10^6 + [2800 \times (0)^2] = 18.2933 \times 10^6 \text{ mm}^4$$

$$I_{xx3} = I_{Gx3} + a_3 h_{y3}^2 = 83333.333 + [1000 \times (145)^2] = 21.0333 \times 10^6 \text{ mm}^4$$

$$I_{xx} = I_{xx1} + I_{xx2} + I_{xx3}$$

$$= 21.033 \times 10^6 + 18.2933 \times 10^6 + 21.0333 \times 10^6 = \boxed{60.36 \times 10^6 \text{ mm}^4}$$

Calculation for I_{yy}

$$h_{x1} = \bar{X} - x_1 = 23.75 - 50 = -26.25 \text{ mm}$$

$$h_{x2} = \bar{X} - x_2 = 23.75 - 5 = 18.75 \text{ mm}$$

$$h_{x3} = \bar{X} - x_3 = 23.75 - 50 = -26.25 \text{ mm}$$

$$I_{Gy1} = I_{Gy3} = \frac{d_1 b_1^3}{12} = \frac{10 \times 100^3}{12} = 0.8333 \times 10^6 \text{ mm}^4$$

$$I_{Gy2} = \frac{d_2 b_2^3}{12} = \frac{280 \times 10^3}{12} = 23.333 \times 10^3 \text{ mm}^4$$

$$I_{yy1} = I_{Gy1} + a_1 h_{x1}^2 = 0.8333 \times 10^6 + [1000 \times (-26.25)^2] = 1.5224 \times 10^6 \text{ mm}^4$$

$$I_{yy2} = I_{Gy2} + a_2 h_{x2}^2 = 23.333 \times 10^3 + [2800 \times (18.75)^2] = 1.0077 \times 10^6 \text{ mm}^4$$

$$I_{yy3} = I_{yy1} = 1.5224 \times 10^6 \text{ mm}^4$$

$$I_{yy} = I_{yy1} + I_{yy2} + I_{yy3}$$

$$= 1.5224 \times 10^6 + 1.0077 \times 10^6 + 1.5224 \times 10^6 = \boxed{4.0525 \times 10^6 \text{ mm}^4}$$

Calculation for radius of gyration

$$K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{60.36 \times 10^6}{4800}} = \boxed{112.138 \text{ mm}}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{\Sigma a}} = \sqrt{\frac{4.0525 \times 10^6}{4800}} = \boxed{29.056 \text{ mm}}$$

Result :
1) $I_{zz} = 60.36 \times 10^6$ **mm^4**
2) $I_{yy} =$ **4.0525 $\times 10^6$ mm^4**
3) $K_{zz} =$ **112.138 mm**
4) $K_{yy} =$ **29.056 mm**

Example : 5.13

Find the moment of inertia of the section shown in the fig.P5.13 about the horizontal centroidal axis. Also find the radius of gyration about that axis.

Unit – III □ P5.14

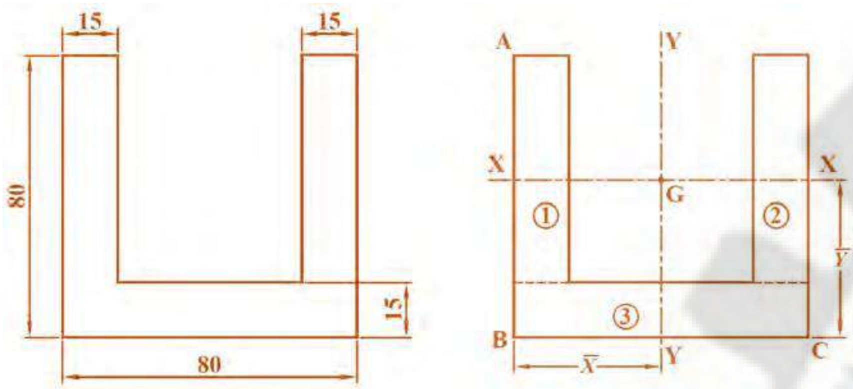


Fig.P5.13 M.I of channel section [Example 5.13]

Solution :

$a_1 = a_2 = 15 \times (80 - 15) = 975\ mm^2$; $a_3 = 80 \times 15 = 1200\ mm^2$

$y_1 = y_2 = 15 + \frac{65}{2} = 47.5\ mm$; $y_3 = \frac{15}{2} = 7.5\ mm$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{(975 \times 47.5) + (1200 \times 7.5) + (975 \times 47.5)}{975 + 1200 + 975}$$

$$= \frac{101625}{3150} = 32.262\ mm$$

Calculation for I_{zz}

$h_{y1} = h_{y2} = \bar{Y} - y_1 = 32.262 - 47.5 = -15.238\ mm$

$h_{y3} = \bar{Y} - y_3 = 32.262 - 7.5 = 24.762\ mm$

$I_{Gx1} = I_{Gx2} = \frac{b_1 d_1^3}{12} = \frac{15 \times 65^3}{12} = 343281.25\ mm^4$

$I_{Gx3} = \frac{b_3 d_3^3}{12} = \frac{80 \times 15^3}{12} = 22500\ mm^4$

$I_{xx1} = I_{Gx1} + a_1 h_{y1}^2$

$= 343281.25 + [975 \times (-15.238)^2] = 569672.978\ mm^4$

$I_{xx2} = I_{xx1} = 569672.978$ mm4

$I_{xx3} = I_{Gx3} + a_3 h_{y3}^2 = 22500 + [1200 \times (24.762)^2] = 1758287.973\ mm^4$

$I_{xx} = I_{xx1} + I_{xx2} + I_{xx3}$

$= 569672.978 + 1758287.973 + 569672.978$

$= 1.8976 \times 10^6\ mm^4$

Radius of gyration, $K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{1.8976 \times 10^6}{3150}} = 25.544$ **mm**

Result : 1) $I_{zz} = 1.8976 \times 10^6\ mm^4$ 2) K_{zz} = **24.544 mm**

Example : 5.14

(Apr.02, Oct.13)

Determine the moment of inertia about centroidal co–ordinate axes of an I–section having equal flanges 120mm × 20mm size and web 120mm × 20mm thick. Also find K_{zz} and K_{yy}.

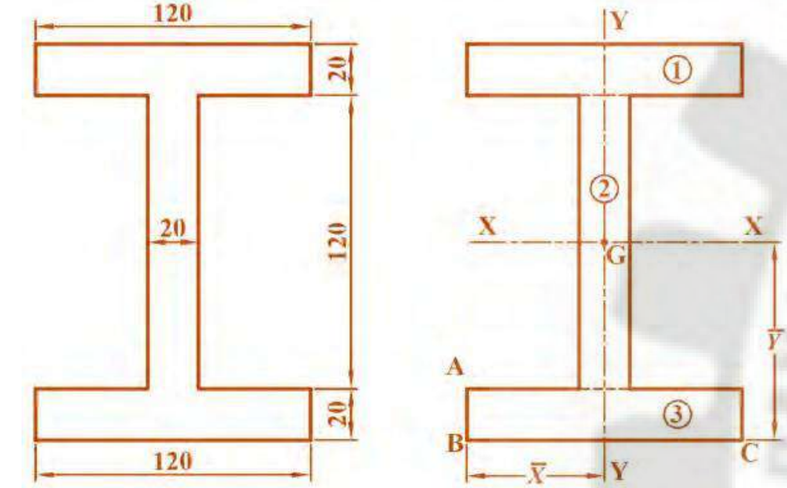


Fig.P5.14 M.I of 'I' section [Example 5.14]

Solution :

This section is symmetrical about X–X and Y-Y axis.

$\therefore \bar{X} = \frac{120}{2} = 60$ mm; $\bar{Y} = \frac{160}{2} = 80$ mm

$a_1 = a_3 = 120 \times 20 = 2400$ mm^2; $a_2 = 20 \times 120 = 2400$ mm^2

$x_1 = x_2 = x_3 = 60$ mm; $y_1 = \frac{20}{2} = 10$ mm; $y_2 = 20 + \frac{120}{2} = 80$ mm;

$y_3 = 20 + 120 + \frac{20}{2} = 150$ mm; $\Sigma a = 2400 + 2400 + 2400 = 7200$ mm^2

Calculation for I_{zz}

$h_{y1} = \bar{Y} - y_1 = 80 - 10 = 70$ mm

$h_{y2} = \bar{Y} - y_2 = 80 - 80 = 0$ mm

$h_{y3} = \bar{Y} - y_3 = 80 - 150 = -70$ mm

$I_{Gx1} = I_{Gx3} = \frac{b_1 d_1^3}{12} = \frac{120 \times 20^3}{12} = 80000$ mm^4

$I_{Gx2} = \frac{b_2 d_2^3}{12} = \frac{20 \times 120^3}{12} = 2.88 \times 10^6$ mm^4

$I_{xx1} = I_{Gx1} + a_1 h_{y1}^2 = 80000 + [2400 \times (70)^2] = 11.84 \times 10^6$ mm^4

$I_{xx2} = I_{Gx2} + a_2 h_{y2}^2 = 2.88 \times 10^6 + [2400 \times 0^2] = 2.88 \times 10^6\ mm^4$

$I_{xx3} = I_{Gx3} + a_3 h_{y3}^2 = 80000 + [2400 \times (-70)^2] = 11.84 \times 10^6 mm^4$

$I_{xx} = I_{xx1} + I_{xx2} + I_{xx3}$

$= 11.84 \times 10^6 + 2.88 \times 10^6 + 11.84 \times 10^6$

$=$ $\boxed{26.56 \times 10^6\ mm^4}$

Calculation for I_{yy}

$h_{x1} = h_{x2} = h_{x3} = \bar{X} - x_1 = 60 - 60 = 0\ mm$

$I_{Gy1} = I_{Gy3} = \dfrac{d_1 b_1^3}{12} = \dfrac{20 \times 120^3}{12} = 2.88 \times 10^6 mm^4$

$I_{Gy2} = \dfrac{d_2 b_2^3}{12} = \dfrac{120 \times 20^3}{12} = 80000\ mm^4$

$I_{yy1} = I_{yy3} = I_{Gy1} + a_1 h_{x1}^2 = 2.88 \times 10^6 + 0 = 2.88 \times 10^6\ mm^4$

$I_{yy2} = I_{Gy2} + a_2 h_{x2}^2 = 80000 + 0 = 80000\ mm^4$

$I_{yy} = I_{yy1} + I_{yy2} + I_{yy3}$

$= 2.88 \times 10^6 + 80000 + 2.88 \times 10^6 =$ $\boxed{5.84 \times 10^6\ mm^4}$

Calculation for radius of gyration

$K_{xx} = \sqrt{\dfrac{I_{xx}}{\Sigma a}} = \sqrt{\dfrac{26.56 \times 10^6}{7200}} =$ $\boxed{60.736\ mm}$

$K_{yy} = \sqrt{\dfrac{I_{yy}}{\Sigma a}} = \sqrt{\dfrac{5.84 \times 10^6}{7200}} =$ $\boxed{28.480\ mm}$

Result : 1) $I_{zz} = 26.56 \times 10^6\ mm^4$ 2) I_{yy} = 5.84×10^6 **mm**^{4}
3) K_{zz} = **60.736 mm** 3) K_{yy} = **28.480 mm**

Example : 5.15 ***(Apr.04, Apr.15, Oct.17)***

An I–section has the top flange 100mm × 15mm, web 150mm × 20mm and the bottom flange 180mm × 30mm. Calculate I_{zz} and I_{yy} of the section. Also find K_{zz} and K_{yy} of the section.

Solution :

This section is symmetrical about Y-Y axis.

$\therefore \bar{X} = \dfrac{180}{2} = 90\ mm$

$a_1 = 180 \times 30 = 5400\ mm^2;\quad y_1 = \dfrac{30}{2} = 15\ mm$

$a_2 = 20 \times 150 = 3000\ mm^2;\quad y_2 = 30 + \dfrac{150}{2} = 105\ mm$

$a_3 = 100 \times 15 = 1500\ mm^2;\quad y_3 = 30 + 150 + \dfrac{15}{2} = 187.5\ mm$

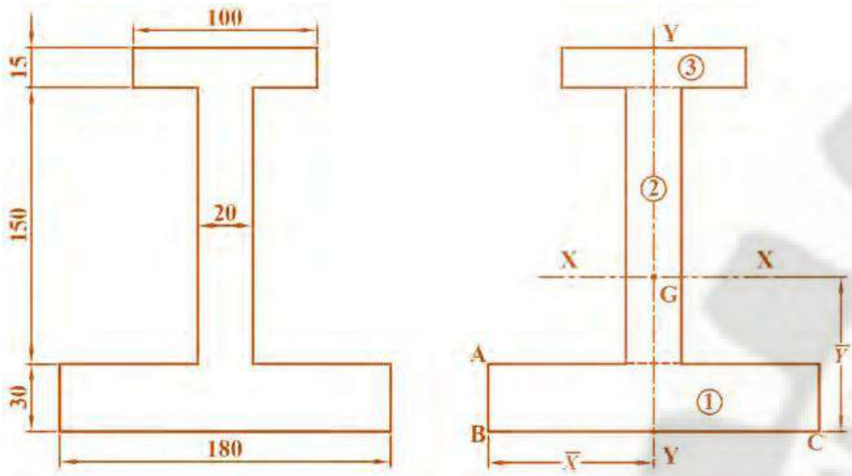


Fig.P5.15 M.I. of 'I' section [Example 5.15]

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(5400 \times 15) + (3000 \times 105) + (1500 \times 187.5)}{5400 + 3000 + 1500}$$

$$= \frac{677250}{9900} = 68.41 \text{ mm}$$

Calculation for I_{zz}

$$h_{y1} = \bar{Y} - y_1 = 68.41 - 15 = 53.41 \text{ mm}$$

$$h_{y2} = \bar{Y} - y_2 = 68.41 - 105 = -36.59 \text{ mm}$$

$$h_{y3} = \bar{Y} - y_3 = 68.41 - 187.5 = -119.09 \text{ mm}$$

$$I_{Gx1} = \frac{b_1 d_1^3}{12} = \frac{180 \times 30^3}{12} = 0.405 \times 10^6 \text{ mm}^4$$

$$I_{Gx2} = \frac{b_2 d_2^3}{12} = \frac{20 \times 150^3}{12} = 5.625 \times 10^6 \text{ mm}^4$$

$$I_{Gx3} = \frac{b_3 d_3^3}{12} = \frac{100 \times 15^3}{12} = 28125 \text{ mm}^4$$

$$I_{xx1} = I_{Gx1} + a_1 h_{y1}^2 = 0.405 \times 10^6 + [5400 \times (53.41)^2] = 15.809 \times 10^6 \text{ mm}^4$$

$$I_{xx2} = I_{Gx2} + a_2 h_{y2}^2 = 5.625 \times 10^6 + [3000 \times (-36.59)^2] = 9.6415 \times 10^6 \text{ mm}^4$$

$$I_{xx3} = I_{Gx3} + a_3 h_{y3}^2 = 28125 + [1500 \times (-119.09)^2] = 21.302 \times 10^6 \text{ mm}^4$$

$I_{xx} = I_{xx1} + I_{xx2} + I_{xx3}$

$= 15.809 \times 10^6 + 9.6415 \times 10^6 + 21.302 \times 10^6 =$ $\boxed{46.7525 \times 10^6 \text{ mm}^4}$

Calculation for I_{yy}

$h_{x1} = h_{x2} = h_{x3} = \bar{X} - x_1 = 90 - 90 = 0$ mm

$$I_{Gy1} = I_{Gy3} = \frac{d_1 b_1^3}{12} = \frac{30 \times 180^3}{12} = 14.58 \times 10^6 \text{mm}^4$$

$$I_{Gy2} = \frac{d_2 b_2^3}{12} = \frac{150 \times 20^3}{12} = 0.1 \times 10^6 \text{ mm}^4$$

$$I_{Gy3} = \frac{d_3 b_3^3}{12} = \frac{15 \times 100^3}{12} = 1.25 \times 10^6 \text{ mm}^4$$

$I_{yy1} = I_{Gy1} + a_1 h^2_{x1} = 14.58 \times 10^6 + 0 = 14.58 \times 10^6 \text{ mm}^4$

$I_{yy2} = I_{Gy2} + a_2 h^2_{x2} = 0.1 \times 10^6 + 0 = 0.1 \times 10^6 \text{ mm}^4$

$I_{yy3} = I_{Gy3} + a_3 h^2_{x3} = 1.25 \times 10^6 + 0 = 1.25 \times 10^6 \text{ mm}^4$

$I_{yy} = I_{yy1} + I_{yy2} + I_{yy2}$

$= 14.58 \times 10^6 + 0.1 \times 10^6 + 1.25 \times 10^6 =$ $\boxed{15.93 \times 10^6 \text{ mm}^4}$

Calculation for radius of gyration

$$K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{46.7525 \times 10^6}{9900}} = \boxed{68.72 \text{ mm}}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{\Sigma a}} = \sqrt{\frac{15.93 \times 10^6}{9900}} = \boxed{40.119 \text{ mm}}$$

Result : 1) $I_{zz} = 46.7525 \times 10^6$ **mm^4**	2) $I_{yy} = 15.85 \times 10^6$ **mm^4**
3) K_{zz} = **68.72 mm**	4) K_{yy} = **40.119 mm**

Example : 5.16

(Oct.01)

A rectangular hole of breadth 60mm and depth 100mm is made at the centre of rectangular plate of breadth 120mm and depth 200mm. Determine the moment of inertia of the hollow plate about its centroidal axis. Also find K_{zz} and K_{yy}.

Solution :

$a_1 = 120 \times 200 = 24000 \text{ mm}^2$; $a_2 = 60 \times 100 = 6000 \text{ mm}^2$;

$\Sigma a = a_1 - a_2 = 24000 - 6000 = 18000 \text{ mm}^2$

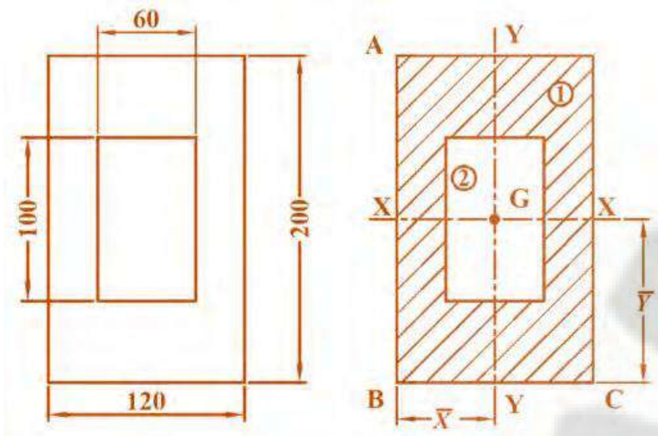


Fig.P5.16 M.I. of hollow rectangular section [Example 5.16]

Calculation for I_{zz}

Moment of inertia of outer rectangle about X-X axis,

$$I_{xx1} = \frac{b_1 d_1^3}{12} = \frac{120 \times 200^3}{12} = 80 \times 10^6 \text{ mm}^4$$

Moment of inertia of inner rectangle about X-X axis,

$$I_{xx2} = \frac{b_2 d_2^3}{12} = \frac{60 \times 100^3}{12} = 5 \times 10^6 \text{ mm}^4$$

Moment of inertia of the whole section about X–X axis,

$$I_{xx} = I_{xx1} - I_{xx2} = 80 \times 10^6 - 5 \times 10^6 = \boxed{75 \times 10^6 \text{ mm}^4}$$

Calculation for I_{yy}

Moment of inertia of outer rectangle about Y–Y axis,

$$I_{yy1} = \frac{d_1 b_1^3}{12} = \frac{200 \times 120^3}{12} = 28.8 \times 10^6 \text{mm}^4$$

Moment of inertia of inner rectangle about Y–Y axis,

$$I_{yy2} = \frac{d_2 b_2^3}{12} = \frac{100 \times 60^3}{12} = 1.8 \times 10^6 \text{ mm}^4$$

Moment of inertia of the whole section about Y–Y axis,

$$I_{yy} = I_{yy1} - I_{yy2} = 28.8 \times 10^6 - 1.8 \times 10^6 = \boxed{27 \times 10^6 \text{ mm}^4}$$

Calculation for radius of gyration

$$K_{xx} = \sqrt{\frac{I_{xx}}{\Sigma a}} = \sqrt{\frac{75 \times 10^6}{18000}} = \boxed{64.55 \text{ mm}}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{\Sigma a}} = \sqrt{\frac{27 \times 10^6}{18000}} = \boxed{38.73 \text{ mm}}$$

Result : 1) $I_{zz} = 75 \times 10^6$ **mm**^{4} 2) I_{yy} **=** 27×10^6 **mm**^{4}

3) K_{zz} = **64.55 mm ;** 4) K_{yy} = **38.73 mm**

Unit - III

Chapter 6. THIN CYLINDERS AND THIN SPHERICAL SHELLS

1. Introduction

Some engineering components like pipes, steam boilers, liquid storage tanks and compressed air reservoirs have greater strength by virtue of their curved shape more than the material by which they are made. These are called *shells*. Generally the walls of such shells are very thin and compared to their diameter. Shells having cylindrical and spherical shapes are widely used. Whenever a shell is subjected to an internal pressure, its walls are subjected to tensile stresses. The shell wall will behave as a membrane in which the stresses are tangential to the middle surface of the wall uniformly distributed across its thickness.

1. Comparison of thin and thick cylindrical shells.

	Thin cylindrical shell	Thick cylindrical shell
1)	The thickness of this cylindrical shell is less than 1/10 to 1/15 times of its diameter.	The thickness of this cylindrical shell is greater than 1/15 times of its diameter.
2)	The normal stresses are assumed to be uniformly distributed throughout the wall thickness	The normal stresses are not uniformly distributed.
3)	Longitudinal stress is uniformly distributed	Longitudinal stress is not uniformly distributed.
4)	The radial stress induced is very small and is neglected.	A finite value of radial stress is induced.

6.3 Assumptions made in design of thin cylindrical shells

The following assumptions are made while designing thin cylindrical shells.

1) The normal stress distribution over a cross section is uniform.
2) Radial stress is small and hence neglected.
3) Loading is assumed to be uniform by neglecting the self weight of the shell.

4) Cylindrical shell is assumed to be subjected to an internal pressure above the atmospheric pressure.

5) Degradation of wall due to corrosion and chemical reaction of contents is neglected.

6.4 Failure of thin cylindrical shell due to internal pressure

Whenever a thin cylindrical shell is subjected to an internal pressure, its walls are subjected to tensile stresses. If the tensile stresses exceed … y fail in any one of the following

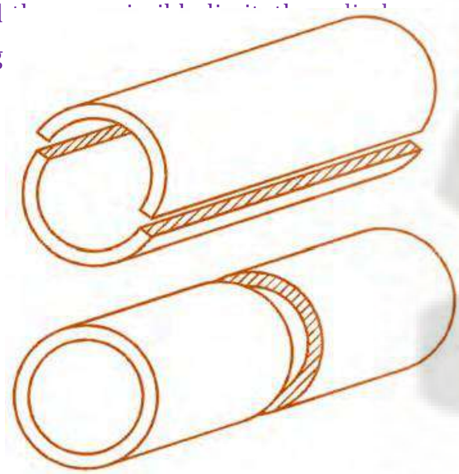


Fig.6.1 Failure of thin cylindrical shell

1) It may split up into two troughs
2) It may split up into two cylinders.

5. Stress in cylindrical shell due to internal pressure

Whenever a thin cylindrical shell is subjected to an internal pressure, its walls will be subjected to the following two types of tensile stresses.

1) Circumferential stress or hoop stress
2) Longitudinal stress

1) Circumferential stress or hoop stress

Consider a thin cylindrical shell subjected to an internal pressure as shown in the fig.6.2. As a result of this pressure, the cylinder may split up in to two troughs.

Let, l = Length of the shell

d = Diameter of the shell

t = Thickness of the shell

p = Intensity of

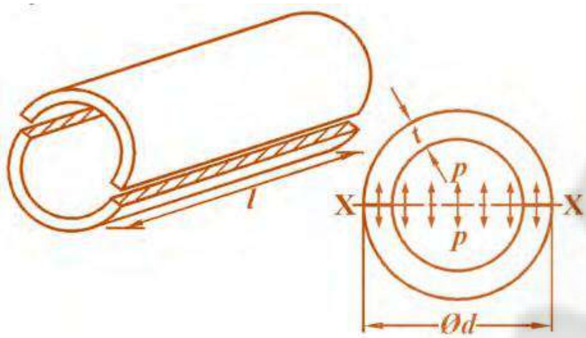


Fig.6.2 Circumferential stress or hoop stress

Let us consider a longitudinal section through the diameter of the shell.

Total force normal to this section

= Intensity of pressure × Projected area

= p × (d × l) = pdl

Resisting force offered by this section

= Circumferential stress × Area of the resisting section

= $f_1(2tl) = 2f_1tl$

Resisting force offered by the section = Total force normal to the section

$$pd\,l = 2f_1tl$$

$$f_1 = \frac{pdl}{2tl} = \frac{pd}{2t}$$

2) Longitudinal stress

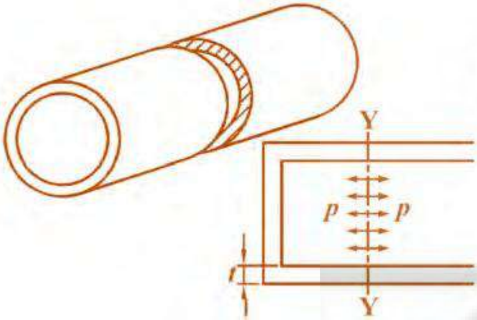


Fig.6.3 Longitudinal stress

Consider a thin cylindrical shell subjected to an internal pressure as shown in the fig.6.3. As a result of this pressure, the cylinder may split up in to two pieces.

Let, l = Length of the shell

d = Diameter of the shell

t = Thickness of the shell

p = Intensity of internal pressure and

f_2 = Longitudinal stress induced in the shell Let us consider a normal section at equilibrium.

The bursting force acts on one end of the shell

= Intensity of pressure× Area

$$= p \times \frac{\pi}{4} d^2$$

Resisting force offered by this section

= Longitudinal stress x Area of the resisting section

$= f_2(\pi dt)$

Resisting force offered by the section = Bursting force acts on one end

$$f_2(\pi dt) = p \times \frac{\pi}{4} d^2$$

$$f_2 = \frac{p \times \frac{\pi}{4} d^2}{\pi d t} = \frac{pd}{4t} = \frac{f_1}{2}$$

6.6. Maximum shear stress

Let f_1 and f_2 be the circumferential stress and longitudinal stress acting at any point on its circumference of a thin cylindrical shell.

The maximum shear stress,

$$f_s = \frac{f_1 - f_2}{2} = \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2} = \frac{pd}{8t}$$

6.7 Changes in dimensions of a thin cylindrical shell due to an internal pressure

Consider a thin shell subjected to an internal pressure.

Let, f_1 = Circumferential or hoop stress which acts in the direction perpendicular to the axis of the cylinder.

f_2 = Longitudinal stress which acts in the direction of length.

e_1 = Circumferential strain

e_2 = Volumetric strain

Y = Volume of cylindrical shell

1/m = Poisson's ratio

$6d$ = Change in diameter of the shell

and

$6l$ = Change in length of the shell

We know that, circumferential strain, $e_1 = \frac{f_1}{E} - \frac{f_2}{mE}$

$$= \frac{1}{E}\left(f_1 - \frac{1}{m}\frac{f_1}{2}\right) \quad \left(\because f_2 = \frac{f_1}{2}\right)$$

$$e_1 = \frac{f_1}{E}\left(1 - \frac{1}{2m}\right)$$

Also circumferential strain, $e_1 = \frac{6d}{d}$

∴ Change in diameter, $6d = e_1 \times d = \frac{f_1}{E}\left(1 - \frac{1}{2m}\right) \times d$

Longitudinal strain, $e_2 = \frac{f_2}{E} - \frac{f_1}{mE}$

$$= \frac{1}{E}\left(\frac{f_1}{2} - \frac{1}{m} f_1\right)$$

$$e_2 = \frac{f_1}{E}\left(\frac{1}{2} - \frac{1}{m}\right)$$

Also, longitudinal strain, $e_2 = \frac{6l}{l}$

∴ Change in length, $6l = e_2 \times l = \boxed{\frac{f_1}{E}\left(\frac{1}{2} - \frac{1}{m}\right) \times l}$

Volume of the cylindrical shell, $Y = \frac{v}{4} d^2 l$

Taking log on both sides,

$$\log Y = \log \frac{v}{4} + \log d^2 + \log l$$

$$\log Y = \log \frac{v}{4} + 2 \log d + \log l$$

Taking differential on both sides,

$$\frac{6Y}{Y} = 0 + 2\frac{6d}{d} + \frac{6l}{l} = 2e_1 + e_2$$

$$= \frac{2 f_1}{E}\left(1 - \frac{1}{2m}\right) + \frac{f_1}{E}\left(\frac{1}{2} - \frac{1}{m}\right)$$

$$= \frac{f_1}{E}\left(2 - \frac{1}{m} + \frac{1}{2} - \frac{1}{m}\right)$$

$$= \frac{f_1}{E}\left(\frac{5}{2} - \frac{2}{m}\right)$$

$$6Y = \frac{f_1}{E}\left(\frac{5}{2} - \frac{2}{m}\right) Y$$

Change in volume,

$$6Y = \frac{f_1}{E}\left(2.5 - \frac{2}{m}\right) Y$$

6.8 Thin spherical shells

Consider a thin spherical shell subjected to an internal pressure as shown in the fig.6.4

Let, p = Intensity of internal pressure

d = Internal diameter of the spherical shell

t = Thickness of the spherical shell

As a result of this internal pressure, the shell is likely to be torn away along the centre of the sphere.

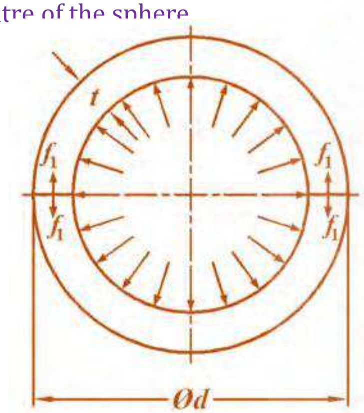


Fig.6.4 Thin spherical shell

Let us consider a section X-X through the centre of the shell.

The bursting force acting along X-X

P = Intensity of internal pressure × Projected area $= p \times \frac{v}{4} d^2$

Let f_1 be the tensile stress induced in the shell at the section X-X.

Resisting force = Tensile stress x Resisting area = $f_1 \times v\, d\, t$

But, resisting force = Bursting force

$$f_1 \times v d t = p \times \frac{v}{4} d^2$$

$$f_1 = \frac{pd}{4t}$$

The tensile stress induced in Y-Y axis, $f_2 = f_1 = \frac{pd}{4t}$

If η is the efficiency of the riveted joint of the spherical shell, then

$$\text{Stress, } f = \frac{pd}{4t\eta}$$

6.9 Change in diameter and volume of thin spherical shell subjected to an internal pressure

Consider a thin spherical shell subjected to an internal pressure as shown in the fig.4.4

Let, p = Intensity of internal pressure

d = Internal diameter of the spherical shell

t = Thickness of the spherical shell

1/m = Poisson's ratio

The tensile stress induced in any direction due to the internal pressure,

$$f_1 = f_2 = f = \frac{pd}{4t}$$

The strain in any direction, $e = e = e = \frac{f_1}{E}\left(1 - \frac{1}{m}\right) = \frac{pd}{4tE}\left(1 - \frac{1}{m}\right)$

Change in diameter, $6d = e \times d = \frac{pd^2}{4tE}\left(1 - \frac{1}{m}\right)$

Original volume of the shell, $Y = \frac{v}{6} d^3$

Taking log on both sides,

$$\log Y = \log \frac{v}{6} + \log d^3 = \log \frac{v}{6} + 3 \log d$$

Taking differential on both sides,

$$\frac{6Y}{Y} = 0 + 3\frac{6d}{d} = 3e = 3 \times \frac{pd}{4tE}\left(1 - \frac{1}{m}\right)$$

Change in volume, $6Y = 3 \times \frac{pd}{4tE}\left(1 - \frac{1}{m}\right) Y$

$$= \frac{3}{1} \times \frac{pd}{4tE}\left(1 - \frac{1}{m}\right) \times \frac{v}{6} d^3$$

Change in volume, $6Y = \frac{vpd^4}{8tE}\left(1 - \frac{1}{m}\right)$

SOLVED PROBLEMS

DETERMINATION OF HOOP STRESS AND LONGITUDINAL STRESS

Example : 6.1 *(Apr.01, Apr.15, Apr.17)*

A boiler 2.8m diameter is subjected to a steam pressure of 0.68 N/mm². Find the hoop stress and longitudinal stresses, if the thickness of the boiler plate is 10mm.

Given : Diameter of boiler, d = 2.8 m = 2800 mm
Internal pressure, p = 0.68 N/mm²
Thickness of the cylinder, t = 10 mm

To find : 1) Hoop stress, f_1 2) Longitudinal stress, f_2

Solution : Hoop stress, $f_1 = \frac{p\,d}{2\,t} = \frac{0.68 \times 2800}{2 \times 10} = 95.2$ N/mm²

Longitudinal stress, $f_2 = \frac{f_1}{2} = \frac{95.2}{2} = 47.6$ N/mm²

Result : 1) Hoop stress, f_1 **= 95.2** N/mm²
2) Longitudinal stress, f_2 **= 47.6** N/mm²

Example : 6.2

A water pipe 1.5m diameter and 15mm wall thickness is subjected to an internal pressure of 1.5 N/mm². Calculate the circumferential and longitudinal stress induced in the pipe.

Given : Diameter of pipe, d = 1.5 m = 1500mm
Wall thickness, t = 15 mm
Internal pressure, p = 1.5 N/mm²

To find : 1) Circumferential stress, f_1 2) Longitudinal stress, f_2

Solution : Circumferential stress, $f_1 = \frac{p\,d}{2\,t} = \frac{1.5 \times 1500}{2 \times 15} = 75$ **N/mm²**

Longitudinal stress, $f_2 = \frac{f_1}{2} = \frac{75}{2} = 37.5$ N/mm²

Result : 1) Circumferential stress, f_1 **= 75** N/mm²
2) Longitudinal stress, f_2 **= 37.5** N/mm²

Example : *(Apr.04)*

A boiler 3m internal diameter is subjected to a boiler pressure of 5bar. Find the hoop and longitudinal stresses, if the thickness of the boiler plate is 14mm.

Given : Diameter of boiler, d = 3 m = 3000 mm

Thickness of plate, t = 10 mm

Steam pressure, p = 5 bar= 5×10^5 N/m^2 = 0.5 N/mm^2

To find : 1) Hoop stress, f_1 2) Longitudinal stress, f_2

Solution :

Hoop stress, $f_1 = \frac{p\,d}{2\,t} = \frac{0.5 \times 3000}{2 \times 10} =$ **75 N/mm^2**

Longitudinal stress, $f_2 = \frac{f_1}{2} = \frac{75}{2} =$ 37.5 N/mm^2

Result : 1) Hoop stress, f_1 **= 75** N/mm^2

2) Longitudinal stress, f_2 **= 37.5** N/mm^2

Example : 6.4 *(Oct.97, Apr.93, Oct.04)*

A gas cylinder of internal diameter 1.5m is 30mm thick. Find the allowable pressure of the gas inside the cylinder if the permissible tensile stress is not to exceed 150 N/mm^2 ***.***

Given : Internal diameter of gas cylinder, d = 1.5m = 1500 mm

Thickness of the gas cylinder, t = 30 mm

Permissible tensile stress = 150 N/mm^2

To find : 1) Allowable pressure of gas inside the cylinder, p

Solution :

Assume the given tensile stress as hoop stress.

We know that, hoop stress, $f_1 = \frac{p\,d}{2\,t}$

$150 = \frac{p \times 1500}{2 \times 30}$

$p = \frac{150 \times 2 \times 30}{1500}$

= 6 **N/mm^2**

Result : Allowable pressure of gas inside the cylinder, p **= 6** N/mm^2

Example : 6.5 *(Oct.03)*

A thin cylindrical shell of 1m diameter is subjected to an internal pressure of 1 N/mm^2 ***. Find the suitable thickness of the shell, if the tensile stress in the material is not to exceed 100*** N/mm^2 ***.***

Given : Diameter of the cylindrical shell, d = 1m = 1000 mm

Internal pressure, p = 1 N/mm^2

Allowable stress = 100 N/mm^2

To find : The thickness of the shell, t

Solution :

Assume the given tensile stress as hoop stress.

We know that, hoop stress, $f_1 = \frac{pd}{2t}$

$$100 = \frac{1 \times 1000}{2 \times t}$$

$$t = \frac{1 \times 1000}{2 \times 100} = 5 \text{ mm}$$

Result : The thickness of the shell, t **= 5mm**

Example :

(Oct.03)

A thin cylindrical shell of 2m diameter is subjected to an internal pressure of 1.5 N/mm^2 ***. Find out the suitable thickness of the ultimate tensile strength of the plate is 500*** N/mm^2 ***. Use a factor of safety of 4.***

Given : Diameter of cylinder, d = 2m = 2000 mm

Internal pressure, p = 1.5 N/mm^2

Ultimate stress = 100 N/mm^2

Factor of safety = 4

To find : 1) The thickness of the shell, t

Solution :

$$\text{Working stress} = \frac{\text{Ultimate stress}}{\text{Factor of safety}} = \frac{500}{4} = 125 \text{ N/mm}^2$$

Assume the given tensile stress as hoop stress.

$$\text{Hoop stress, } f_1 = \frac{p\,d}{2\,t}$$

$$125 = \frac{1.5 \times 2000}{2 \times t}$$

$$t = \frac{1.5 \times 2000}{2 \times 125} = 12 \text{ mm}$$

Result : 1) The thickness of the shell, t **= 12 mm**

Example : 6.7 *(Apr.92)*

A water main 500mm diameter contains water at a pressure head of 100mm. The weight of the water is 10 KN/mm^3 ***. Find the thickness of the metal required if the permissible stress is 25*** N/mm^2 ***.***

Given : Diameter of water main, d = 500 mm

Pressure head, h = 100 m = 100×10^3 mm

Permissible stress, $f_1 = 25\ N/mm^2$

Weight of water, r = $10\ KN/mm^3 = \frac{10 \times 10^3}{10^9}\ N/mm^3$

To find : 1) The thickness of the metal, t

Solution : Internal pressure of water, p = r × h

$$= \frac{10 \times 10^3}{10^9} \times 100 \times 10^3 = 1\ N/mm^2$$

Let the permissible stree be the hoop stress

Hoop stress, $f_1 = \frac{p\,d}{2\,t}$

$$25 = \frac{1 \times 500}{2 \times t}$$

$$t = \frac{1 \times 500}{2 \times 25} = \boxed{10\text{ mm}}$$

Result : 1) The thickness of the metal required, t **= 10 mm**

Example : 6.8 *(Oct.97, Oct.01, Apr.05, Apr.18)*

A long steel tube 70mm internal diameter and wall thickness 2.5mm has closed ends and subjected to an internal pressure of 10 N/mm^2 ***. Calculate the magnitude of hoop stress and longitudinal stresses set up in the tube. If the efficiency of the longitudinal joint is 80%, state the stress which is affected and what is its revised value.***

Given : Diameter of the steel tube, d = 70 mm

Wall thickness, t = 2.5 mm

Internal pressure, p = $10 N/mm^2$

Efficiency of the joint, η = 80 % = 0.8

To find : 1) Hoop stress, f_1 2) Longitudinal stress, f_2

Solution :

Hoop stress, $f_1 = \frac{p\,d}{2\,t} = \frac{10 \times 70}{2 \times 2.5} = 140\ N/mm^2$

Longitudinal stress, $f_2 = \frac{f_1}{2} = \frac{140}{2} = 70\ N/mm^2$

The hoop is affected by the longitudinal joint.

When the efficiency is 0.8,

Revised value of hoop stress, $f_1 = \frac{p\,d}{2\,t\,\eta} = \frac{10 \times 70}{2 \times 2.5 \times 0.8} = 175\ N/mm^2$

Result : 1)Hoop stress, f_1=140 N/mm^2 2)Longitudinal stress, f_2=70 N/mm^2

3) Revised value of hoop stress when the efficiency of longitudinal joint is 80%, f_1 = 175 N/mm^2

DETERMINATION OF CHANGE IN DIMENSIONS OF THIN CYLINDRICAL SHELLS

Example : *(Oct.03, Oct.13, Oct.17)*

A cylindrical shell 3m long and 500 mm in diameter is made up of 20 mm thick plate. If the cylindrical shell is subjected to an internal pressure of 5 N/mm^2***, find the Result :ing hoop stress, longitudinal stress, changes in diameter, length and volume. Take*** $E = 2 \times 10^5\ N/mm^2$ ***and Poisson's ratio = 0.3.***

Given : Length of cylinder, l = 3m = 3000 mm

Internal diameter, d = 500 mm

Metal thickness, t = 20 mm

Internal pressure, p = 5 N/mm^2

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

Poisson's ratio, 1/m = 0.3

To find : 1) Hoop stress, f_1 2) Longitudinal stress, f_2

3) Change in diameter, δd 4) Change in length, δl

5) Change in volume, δV

Solution :

Volume of the shell, $V = \frac{\pi}{4} d^2 l = \frac{\pi}{4} \times 500^2 \times 3000 = 589.0486 \times 10^6\ mm^3$

Circumferential stress, $f_1 = \frac{p\,d}{2\,t} = \frac{5 \times 500}{2 \times 20} = 62.5\ N/mm^2$

The longitudinal stress, $f_2 = \frac{f_1}{2} = \frac{62.5}{2} = 31.25\ N/mm^2$

Circumferential strain, $e = \frac{1}{E}\left[f_1 - \frac{f_2}{m}\right]$

$$= \frac{1}{2 \times 10^5}[62.5 - 0.3 \times 31.25] = \boxed{2.65625 \times 10^{-4}}$$

Longitudinal strain, $e_2 = \frac{1}{E}\left[f_2 - \frac{1}{m} f_1\right]$

$$= \frac{1}{2 \times 10^5}[31.25 - 0.3 \times 62.5] = \boxed{6.25 \times 10^{-5}}$$

Change in diameter, $\delta d = e_1 \times d = 2.65625 \times 10^{-4} \times 500 = \boxed{0.1328 \text{ mm}}$

Change in length, $\delta l = e_2 \times l = 6.25 \times 10^{-5} \times 3000 = \boxed{0.1875 \text{ mm}}$

Change in volume, $\delta V = (2e_1 + e_2) \times V$

$$= (2 \times 2.65625 \times 10^{-4} + 6.25 \times 10^{-5}) \times 589.0486 \times 10^6$$

$$= \boxed{349.748 \times 10^3 \text{ mm}^3}$$

Result :
1) Hoop stress, f_1 **= 62.5** N/mm^2
2) Longitudinal stress, f_2 **= 31.25** N/mm^2
3) Change in diameter, δd **= 0.1328 mm**
4) Change in length, δl **= 0.1875 mm**
5) Change in volume, δV **=** 349.748×10^3 mm^3

Example : 6.10

(Apr. 04, Oct.12, Apr.17)

Calculate the increase in volume of a boiler 3m long and 1.5m diameter, when subjected to an internal pressure of 2 N/mm^2***. The thickness is such that the maximum tensile stress is not to exceed 30*** N/mm^2***. Take*** $E = 2.1 \times 10^5$ N/mm^2 ***and 1/m = 0.28. Also calculate the changes in diameter and length.***

Given : Length of the boiler shell, l = 3m = 3000 mm

Diameter of the boiler shell, d = 1.5 m = 1500 mm

Internal pressure, p = 2 N/mm^2

Maximum tensile stress, f_1 = 30 N/mm^2

Young's modulus, $E = 2.1 \times 10^5$ N/mm^2

Poisson's ratio, 1/m = 0.28

To find : 1) Increase in volume, δV 2) Change in diameter, δd 3) Change in length, δl

Solution :

Longitudinal stress, $f_2 = \frac{f_1}{2} = \frac{30}{2} = 15$ N/mm^2

Volume of the shell, $V = \frac{\pi}{4} \times d^2 l$

$$= \frac{\pi}{4} \times 1500^2 \times 3000 = 5.3014 \times 10^9 \text{ mm}^3$$

Increase in volume, $\delta V = \frac{f_1}{E}\left[2.5 - \frac{2}{m}\right] \times V$

$$= \frac{30}{2.1 \times 10^5}[2.5 - 2 \times 0.28] \times 5.3014 \times 10^9 = \boxed{1.469 \times 10^6 \text{ mm}^3}$$

Change in diameter, $\delta d = \frac{1}{E}\left[f_1 - \frac{f_2}{m}\right] \times d$

$$= \frac{1}{2.1 \times 10^5}[30 - 0.28 \times 15] \times 1500 = \boxed{0.1843 \text{ mm}}$$

Change in length, $\delta l = \frac{1}{E}\left[f_2 - \frac{f_1}{m}\right] \times l$

$$= \frac{1}{2.1 \times 10^5}[15 - 0.28 \times 30] \times 3000 = \boxed{0.0943 \text{ mm}}$$

Result : 1) Increase in volume, 6Y = 1.469×10^6 mm³
2) Change in diameter, 6d = 0.1843 mm
3) Change in length, 6l = 0.0943 **mm**

THIN SPHERICAL SHELLS

Example : 6.11

A vessel in the shape of a thin spherical shell 2m in diameter and 5mm thickness is completely filled with a fluid at a pressure of 0.1 N/mm². Determine the stress induced in the shell material.

Given : Diameter of the shell, d = 2 m = 2000 mm

Thickness of the shell, t = 5 mm

Intensity of pressure, p = 0.1 N/mm²

To find : 1) Tensile stress, f

Solution :

Tensile stress, $f = \frac{pd}{4t} = \frac{0.1 \times 2000}{4 \times 5} = \boxed{10 \text{ N/mm}^2}$

Result : Tensile stress, f = **10** N/mm²

Example : 6.12

A spherical vessel of 3m diameter is subjected to an internal pressure of 1.5 N/mm² ***. Find the thickness of the plate, if the maximum stress is not to exceed 90*** N/mm² ***. The efficiency of the joint is 75%.***

Given : Diameter of spherical shell, d = 3 m = 3000 mm

Internal pressure, p = 1.5 N/mm^2

Tensile stress, f = 90 N/mm^2

Efficiency of the joint, η = 75 % = 0.75

To find : The thickness of the plate, t

Solution :

We know that, tensile stress, $f = \dfrac{pd}{4t\eta}$

$$90 = \frac{1.5 \times 3000}{4 \times t \times 0.75}$$

$$t = \frac{1.5 \times 3000}{90 \times 4 \times 0.75} = \boxed{16.667 \text{ mm}}$$

Result : 1) The thickness of the plate, t **= 16.667 mm**

Example : 6.13

(Oct.01, Oct.18)

Determine the change in diameter and change in volume of spherical shell 2m in diameter and 12mm thick subjected to an internal pressure of 2 N/mm^2. Assume E = 2 × 10^5 N/mm^2 and Poisson's ratio = 0.25.

Given : Diameter of spherical shell, d = 2 m = 2000 mm

Thickness of the shell, t = 12 mm

Internal pressure, p = 2 N/mm^2

Young's modulus, E = 2 × 10^5 N/mm^2

Poisson's ratio, 1/m = 0.25

To find : 1) Change in diameter, δd 2) Change in volume, δV

Solution :

Volume of shell, $V = \dfrac{\pi}{6} \times d^3 = \dfrac{\pi}{6} \times 2000^3 = 4.18879 \times 10^9 \text{ mm}^3$

Strain in the spherical shell, $e = \dfrac{f_1}{E}\left[1 - \dfrac{1}{m}\right] = \dfrac{pd}{4tE}\left[1 - \dfrac{1}{m}\right]$

$$= \frac{2 \times 2000}{4 \times 12 \times 2 \times 10^5}[1 - 0.25] = 3.125 \times 10^{-4}$$

Change in diameter, $\delta d = e \times d = 3.125 \times 10^{-4} \times 2000 = \boxed{0.625 \text{ mm}}$

Change in volume, $\delta V = 3e \times V$

$$= 3 \times 3.125 \times 10^{-4} \times 4.18879 \times 10^9 = \boxed{3.927 \times 10^6 \text{ mm}^3}$$

Result : 1) Change in diameter, δd **= 0.625 mm**

2) Change in volume, δV **= 3.927 × 10^6 mm^3**

Example : 6.14 *(Apr.01, Apr.13)*

Determine the depth to which a spherical float 200mm diameter and 6mm thickness have to be immersed in water in order that its diameter is decreased by 0.05mm. Assume $E = 2 \times 10^5\ N/mm^2$, $1/m$ ***= 0.25 and weight of water = 9810*** N/m^3.

Given : Diameter of float, d = 200 mm

Thickness of float, t = 6 mm

Change in diameter, ðd = 0.05 mm

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

Poisson's ratio, $1/m = 0.25$

Weight of water, $r = 9810\ N/m^3 = 9810 \times 10^{-9}\ N/mm^3$

To find : 1) Depth to which float to be immersed, *h*

Solution : Change in diameter of spherical float,

$$ðd = \frac{pd^2}{4tE}\left[1 - \frac{1}{m}\right]$$

$$0.05 = \frac{p \times 200^2}{4 \times 6 \times 2 \times 10^5}[1 - 0.25]$$

$$p = \frac{0.05 \times 4 \times 6 \times 2 \times 10^5}{200^2} = 8\ N/mm^2$$

We know that, pressure, $p = r \times h$

$$h = \frac{p}{r} = \frac{8}{9810 \times 10^{-9}} = \boxed{815494.394\ mm}$$

Result : 1) Depth to which float to be immersed, h **=815494.394 mm**

Example : 6.15 *(Apr.01, Oct.16)*

A spherical shell of 1m internal diameter and 5mm thick is filled with a fluid until its volume increases by $0.2 \times 10^6\ mm^3$. ***Calculate the pressure exerted by the fluid on the shell. Take*** $E = 2 \times 10^5\ N/mm^2$, $1/m$ ***= 0.3 for the material.***

Given : Internal diameter of spherical shell = 1000 mm

Thickness of spherical shell, t = 5 mm

Increase in volume $ðV = 0.2 \times 10^6\ mm^3$

Young's modulus, E = $E = 2 \times 10^5\ N/mm^2$

Poisson's ratio, $1/m = 0.3$

To find : 1) Pressure exerted by the fluid, p

Solution :

Volume of shell, $V = \frac{\pi}{6} \times d^3 = \frac{\pi}{6} \times 1000^3 = 5.236 \times 10^8 \text{ mm}^3$

Change in volume of spherical shell, $\delta V = 3 \times \frac{pd}{4tE}\left[1 - \frac{1}{m}\right] \times V$

$$0.2 \times 10^6 = \frac{3 \times p \times 1000}{4 \times 5 \times 2 \times 10^5}[1 - 0.3] \times 5.236 \times 10^8$$

$$p = \frac{0.2 \times 10^6 \times 4 \times 5 \times 2 \times 10^5}{3 \times 1000 \times 0.7 \times 5.236 \times 10^8} = 0.7276 \text{ N/mm}^2$$

Result : 1) Pressure exerted by the fluid, p **= 0.7276** N/mm^2

boiler.

2

Unit – IV

Chapter 7. THEORY OF TORSION

1. Introduction

Power is generally transmitted through shafts. While transmitting power, a turning force is applied in a vertical plane perpendicular to the axis of the shaft. The product of this turning force and distance of its application
from the centre of the shaft is known as *torque, turning moment* or *twisting moment*. A shaft of a circular section is said to be in torsion when it is subjected to torque.

1. Pure torsion

A circular shaft is said to be in a state of pure torsion when it is subjected to pure torque and not accompanied by any other force such as
bending or axial force. Due to this torsion, the state of stress at any point in the cross–section is one of pure shear. The shearing stress thus induced in the shaft produces a moment of resistance, equal and opposite to the applied
torque.

1. Assumption made in theory of pure torsion

The following assumptions are made in the theory of pure torsion which relates shear stress and the angle of twist to the applied torque.

1) The material of the shaft is uniform throughout.
2) The material of the shaft obeys Hooke's law.
3) The shaft is of uniform circular section throughout.
4) The shaft is subjected to twisting couples whose planes are exactly perpendicular to the longitudinal axis.
5) The twist along the shaft is uniform.
6) Stresses do not exceed the limit of proportionality.
7) All diameters which are straight before twist remain straight after twist.
8) Normal cross–sections at the shaft, which were plane and

7.4 Derivation of torsion equation

a) To prove $\frac{f_s}{r} = \frac{C\&}{l}$

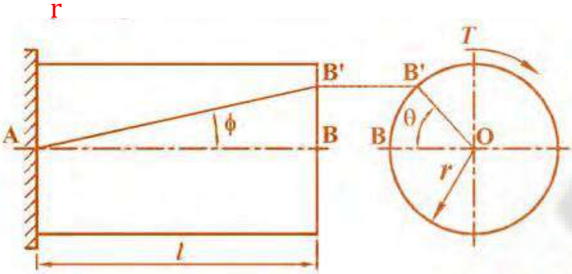


Fig.7.1 Shaft under pure torsion

Consider a shaft fixed at one end and subjected to a torque at the other end as shown in the fig.7.1.

Let, T = Torque

l = Length of the shaft

r = Radius of circular shaft

As a result of the torque, every cross section of the shaft is subjected to shear stresses. Let the line AB on the surface of the shaft be deformed to AB' and OB to OB' as shown in the fig.

Let, ∠BAB' = ø in degrees

∠BOB' = & in radians

f_s = Shear stress induced in the surface

C = Modulus of rigidity of the shaft material.

We know that,

$$\text{Shear strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{BB'}{l} = \tan ø = ø \quad \text{------- (1)}$$

Since ϕ is very small, tan ø = ø

We also know that, arc BB' = r&

$$ø = \frac{BB'}{l} = \frac{r\&}{l} \quad \text{------- (2)}$$

If f_s is the intensity of shear stress on the outermost layer, then

$$\text{Modulus of rigidity, } C = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{f_s}{ø}$$

$$ø = \frac{f_s}{C} \quad \text{------- (3)}$$

$$\text{Equating (2) and (3)} \Rightarrow \frac{r\&}{l} = \frac{f_s}{C}$$

(or)

$$\boxed{\frac{f_s}{r} = \frac{C\&}{l}}$$

Since &, C and l are constants, the intensity of stress at any section of the shaft is proportional to the distance of the point from the axis of the shaft.

i.e. $\frac{f_1}{r_1} = \frac{f_2}{r_2} = \frac{f_s}{r}$

b) To prove $\frac{T}{J} = \frac{C\&}{l} = \frac{f_s}{r}$

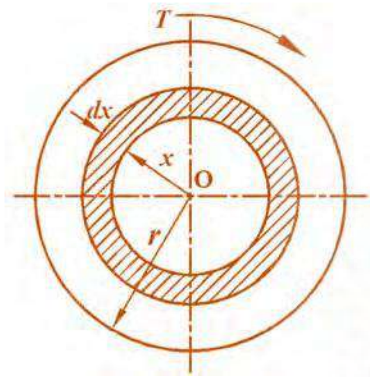


Fig.7.2 Shaft under pure torsion

Consider a shaft subjected to torque T as shown in the fig.7.2

Consider an elemental area 'da' of thickness 'dx' at a distance 'x' from the centre of the shaft.

Let, r = Radius of the shaft and

f_s = Shear stress developed in the outermost layer of the shaft.

Shear stress at this section $= f_s \times \frac{x}{r}$

Area of the elemental strip, da = 2лx × dx

Turning force on the elemental area = Shear stress × Area

$= f_s \frac{x}{r} \times 2\text{л}x\, dx$

$= \frac{2\text{л}}{r} \times f_s (x^2 dx)$

Turning moment (torque) of this element,

dT = Shear force × Distance of element from axis

$= \frac{2\text{л}}{r} f_s (x^2 dx) \times x = \frac{2\text{л}}{r} f_s x^3 dx$

Total torque can be found out by integrating the above equation between '0' and 'r'.

$$T = \int_0^r \frac{2\pi f_s}{r} x^3 \, dx = \frac{2\pi f_s}{r}\left[\frac{x^4}{4}\right]_0^r = \frac{2\pi}{r} f_s \times \frac{r^4}{4}$$

$$T = \frac{\pi}{2} f_s r^3 = \frac{\pi}{16} f_s d^3 \qquad \because r = \frac{d}{2}$$

$$f_s = \frac{16T}{\pi d^3}$$ ---------(1)

We know that, $\frac{f_s}{r} = \frac{C\&}{l}$ --------(2)

Substituting the value of f_s in equation (2)

$$\frac{16T}{\pi d^3 \times \frac{d}{2}} = \frac{C\&}{l} \Rightarrow \frac{T}{\frac{\pi}{32} d^4} = \frac{C\&}{l}$$

$$\frac{T}{J} = \frac{C\&}{l}$$ ---------(3)

Where, $J = \frac{\pi}{32} d^4$ which is known as ***polar moment of inertia***

Combining equation (2) and (3) ⇒ $\frac{T}{J} = \frac{f_s}{r} = \frac{C\&}{l}$

The above relation can be rewritten as ⇒ $\frac{T}{J} = \frac{f_s}{r}$; $\frac{T}{J} = \frac{C\&}{l}$

7.5 Strength of hollow shaft

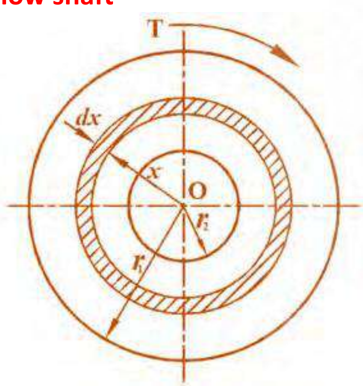


Fig.7.3 Hollow circular shaft subjected to pure torsion

Consider a hollow shaft subjected to toque 'T' as shown in the fig.7.3. Let r_1 and r_2 be the outside and inside radius of the hollow shaft respectively. Let us consider an elemental area 'da' at distance 'x ' from the centre of the shaft and of thickness 'dz' as shown in the fig.

Area of the elemental strip, $da = 2\pi x \, dx$.

Shear stress at this section, $f_x = f_s \frac{x}{r_1}$

Turning force = Stress × Area = $f_s \frac{x}{r_1} 2\pi x \, dx = \frac{f_s}{r_1} 2\pi x^2 dx$

Turning moment (torque) of this element,

dT = Shear force × Distance of element from axis

$$= \frac{2\pi}{r_1} f_s x^2 dx \cdot x = \frac{2\pi}{r_1} f_s x^3 dx$$

Total torque can be found out by integrating the above equation between r_2 and r_1.

$$T = \int_{r_2}^{r_1} \frac{2\pi f_s}{r_1} x^3 dx = \frac{2\pi f_s}{r_1}\left[\frac{x^4}{4}\right]_{r_2}^{r_1}$$

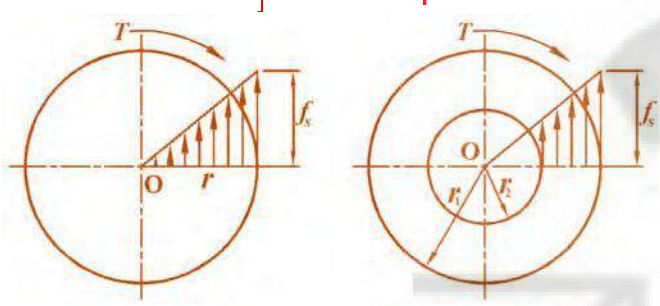
$$= \frac{2\pi f_s}{r_1}\left[\frac{r_1^4 - r_2^4}{4}\right]$$

$$= \frac{2\pi f_s}{(d_1/2)}\left[\frac{(d_1/2)^4 - (d_2/2)^4}{4}\right]$$

$$= \frac{4\pi f_s}{d_1}\left[\frac{d_1^4 - d_2^4}{4 \times 16}\right]$$

$$T = \frac{\pi}{16} f_s \left[\frac{d_1^4 - d_2^4}{d_1}\right]$$

7.6 Stress distribution in the shaft under pure torsion



(i) Solid shaft (ii) Hollow shaft

Fig.7.4 Shear stress distribution

The intensity of shear stress at any point in the cross–section of a shaft subjected to pure torsion is proportional to its distance from the centre. In other words, the shear intensity is zero at the axis of the shaft and increases linearly to maximum of f_s at the surface. The shear stress at any point on the circumference is same. The intensity of shear stress in hollow shaft is more or less uniform throughout the section.

7.7 Power transmitted by the shaft

Consider a rotating shaft which transmits power from one of its ends to another.

Let, N = Speed of the shaft in rpm and

T = Average torque in KN–m

Work done per minute = Force × Distance

$= T \times 2\pi N = 2\pi N T$

$\therefore$ Work done per second $= \frac{2\pi N T}{60}$ (KN –m)

Power transmitted = Work done per second

$$P = \frac{2\pi N T}{60} \quad \textbf{(KW)}$$

7.8 Polar modulus

The ratio between the polar moment of inertia of the cross–section of the shaft and the maximum radius of the section is known as polar modulus or polar section modulus. It is an important parameter, generally used in the design of shaft. It is denoted by Z.

$$Z = \frac{\text{Polar moment of inertia}}{\text{Maximum radius}} = \frac{J}{r}$$

For a solid circular shaft, $J = \frac{\pi}{32} d^4$; $r = \frac{d}{2}$

$$Z = \frac{J}{r} = \frac{\frac{\pi}{32} d^4}{(d/2)} = \frac{\pi}{16} d^3$$

For a hollow circular shaft, $J = \frac{\pi}{32}(d_1^{\,4} - d_2^{\,4})$; $r = \frac{d_1}{2}$

$$Z = \frac{J}{r} = \frac{\frac{\pi}{32}(d_1^{\,4} - d_2^{\,4})}{(d_1/2)} = \frac{\pi (d_1^{\,4} - d_2^{\,4})}{16\, d_1}$$

7.9 Torsional strength

It is defined as the torque developed per unit maximum shear stress.

Torsional strength is also known as the *efficiency* of a shaft.

$$\text{Torsional strength} = \frac{T}{f_s}$$

From the equation $\frac{T}{J} = \frac{f_s}{r}$

$$\frac{T}{f_s} = \frac{J}{r} = Z$$

Therefore, torsional strength may also be represented by the section modulus. For a given material and weight, a hollow shaft withstands larger value of torque when compared to that of solid shaft. Because for a given cross–sectional area, hollow circular section has larger section modulus when compared to that of solid circular section.

7.10 Torsional rigidity or stiffness

Torsional rigidity or stiffness is defined as the torque required to produce an unit angle of twist in a specified length of the shaft.

$$\text{Torsional rigidity} = \frac{T}{\&}$$

From the equation $\frac{T}{J} = \frac{C\&}{l}$

$$\frac{T}{\&} = \frac{CJ}{l}$$

From the above equation it is evident that torsional rigidity or stiffness is the product of modulus of rigidity and polar moment of inertia over a unit length of the shaft. For a given cross–sectional area, torsional rigidity of a hollow circular shaft is larger when compared to that of solid circular shaft.

7.11 Comparison of hollow shaft and solid shaft

Let, d = Diameter of the solid shaft

d_1 = Outside diameter of the hollow shaft

d_2 = Inside diameter of the hollow shaft

a) Comparison by strength consideration

$$\frac{\text{Strength of the hollow shaft}}{\text{Strength of the solid shaft}} = \frac{\text{Section modulus of hollow shaft}}{\text{Section modulus of solid shaft}}$$

$$= \frac{\frac{\pi}{16 d_1}(d_1^4 - d_2^4)}{\frac{\pi d^3}{16}} = \frac{(d_1^4 - d_2^4)}{d_1 \times d^3}$$

For a given cross–sectional area a hollow circular shaft has larger value of section modulus when compared with that of a solid circular shaft. So the hollow shaft has more strength than that of a solid shaft.

b) Comparison by weight consideration

Let, l = Length of both the solid and hollow shaft

ρ = Density of both the material of solid and hollow shaft

A_s = Cross–sectional area of the solid shaft

A_h = Cross–sectional area of the hollow shaft

Weight of the solid shaft, W_s = Density × Volume

$$= \rho \times l \times A_s = \rho\, l\, \frac{\pi}{4} d^2$$

Weight of the hollow shaft, W_h = Density × Volume

$$= \rho \times l \times A_h = \rho\, l\, \frac{\pi}{4}(d_1^2 - d_2^2)$$

$$\frac{\text{Weight of the solid shaft}}{\text{Weight of the hollow shaft}} = \frac{\rho\, l\, \frac{\pi}{4} d^2}{\rho\, l\, \frac{\pi}{4}(d_1^2 - d_2^2)} = \frac{d^2}{(d_1^2 - d_2^2)}$$

For a given material, length and torsional strength, the weight of a hollow shaft is less than that of a solid shaft. When using hollow shaft, the material requirement is considerably reduced.

$$\textbf{\% Saving in material} = \frac{W_s - W_h}{W_s} \times 100 = \frac{A_s - A_h}{A_s} \times 100$$

7.12 Advantages of hollow shaft over solid shaft

The following are the advantages of hollow shaft over solid shaft.

1) A hollow shaft has greater torsional strength than a solid shaft of same material.
2) A hollow shat has more stiffness than a solid shaft of same cross– sectional area.
3) The material required for hollow shaft is comparatively lesser than the solid shaft for same strength.
4) Hollow shaft is lighter in weight than a solid shaft of equal strength.
5) The removal of core from large shafts increase their reliability.
6) The material in the hollow shaft is effectively utilized.
7) The shear stress induced in the hollow shaft is almost uniform throughout the section.

SOLVED PROBLEMS

Example : 7.1 *(Apr.01)*

***Calculate the torque in a solid circular shaft 120mm diameter, if the shear stress is not to exceed 80 N/mm²*.**

Given : Diameter of shaft, d = 120 mm

Maximum shear stress, $f_s = 80\ N/mm^2$

To find : 1) Torque, T

Solution :

Torque in a solid circular shaft,

$$T = \frac{\pi}{16} f_s d^3 = \frac{\pi}{16} \times 80 \times 120^3 = \mathbf{27.143 \times 10^6\ N\text{-}mm}$$

Result : 1) Torque in the shaft, $T = 27.143 \times 10^6$ **N-mm**

Example : 7.2

A solid steel shaft is to transmit a torque of 10KN–m. If the shearing stress is not to exceed 45 N/mm², find the minimum diameter of the shaft.

Given : Torque, T = 10 KN–m = 10×10^6 N–mm

Maximum shearing stress, $f_s = 45\ N/mm^2$

To find : 1) Minimum diameter of shaft, d

Solution :

Torque in a solid circular shaft, $T = \frac{\pi}{16} f_s d^3$

$$d^3 = \frac{16 \times T}{\pi \times f_s} = \frac{16 \times 10 \times 10^6}{\pi \times 45} = 1.13177 \times 10^6$$

d = **104 mm**

Result : 1) Minimum diameter of the shaft, d = **104 mm**

Example : 7.3

A hollow shaft of external and internal diameter of 80mm and 50mm is required to transmit torque from one end to the other. What is the safe torque it can transmit, if the allowable shear stress is 45 N/mm²?

Given : External diameter of the shaft, $d_1 = 80$ mm

Inter diameter of the shaft, $d_2 = 50$ mm

Allowable shear stress, $f_s = 45\ N/mm^2$

To find : 1) Torque transmitted by the shaft, T

Solution :

Torque transmitted by the hollow circular shaft,

$$T = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi}{16} \times 45 \times \frac{80^4 - 50^4}{80}$$

$$= \boxed{3.834 \times 10^6 \text{ N-mm}}$$

Result : 1) Torque transmitted by the shaft, T = 3.834×10^6 **N-mm**

Example : 7.4

(Oct.12, Apr.15, Apr.17)

Calculate the power transmitted by a shaft 100 mm diameter running at 250 rpm, if the shear stress in the shaft material is not to exceed 75 N/mm^2.

Given : Diameter of the shaft, d = 100 mm

Speed of the shaft, N = 250 rpm

Maximum shear stress, f_s = 75 N/mm^2

To find : 1) Power transmitted by the shaft, P

Solution :

Torque transmitted by the shaft,

$$T = \frac{\pi}{16} f_s d^3 = \frac{\pi}{16} \times 75 \times 100^3 = 14.726 \times 10^6 \text{ N-mm} = 14.726 \text{ KN-m}$$

Power transmitted by the shaft,

$$P = \frac{2\pi N T}{60} = \frac{2 \times \pi \times 250 \times 14.726}{60} = \boxed{385.53 \text{ KW}}$$

Result : 1) The power transmitted by the shaft, P = **385.53 KW**

Example : 7.5

(Oct.13)

A hollow shaft of external and internal diameters as 100mm and 40mm is transmitting power at 120 rpm. Find the power the shaft can transmit, if the shearing stress is not to exceed 50 N/mm^2.

Given : External diameter of the shaft, d_1 = 100 mm

Inter diameter of the shaft, d_2 = 40 mm

Speed of the shaft, N = 120 rpm

Allowable shear stress, f_s = 50 N/mm^2

To find : 1) Power transmitted by the shaft, P

Solution :

Torque transmitted by the hollow circular shaft,

$$T = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi}{16} \times 50 \times \frac{100^4 - 40^4}{100}$$

$= 9.566 \times 10^6$ N-mm = 9.566 KN-m

Power which can be transmitted by the shaft,

$$P = \frac{2\pi N T}{60} = \frac{2 \times \pi \times 120 \times 9.566}{60} = \boxed{120.21 \text{ KW}}$$

Result : 1) Power transmitted by the shaft, P = **120.21 KW**

Example : 7.6

A solid circular shaft of 100mm diameter is transmitting 120KW at 150 rpm. Find the intensity of shear stress in the shaft.

Given : Diameter of the shaft, d = 100 mm

Power transmitted, P = 120 KW

Speed of the shaft, N = 150 rpm

To find : 1) Intensity of shear stress, f_s

Solution :

Power transmitted by the shaft,

$$P = \frac{2\pi N T}{60}$$

$$T = \frac{P \times 60}{2 \times \pi \times N} = \frac{120 \times 60}{2 \times \pi \times 150} = 7.639 \text{ KN-m} = 7.639 \times 10^6 \text{ N-mm}$$

Also, torque transmitted by the shaft,

$$T = \frac{\pi}{16} f_s d^3$$

$$f_s = \frac{16 \times T}{\pi \times d^3} = \frac{16 \times 7.639 \times 10^6}{\pi \times 100^3} = \boxed{38.905 \text{ N/mm}^2}$$

Result : 1) Intensity of shear stress, f_s = **38.905** N/mm^2

Example : 7.7

(Oct.17)

A hollow circular shaft of 25 mm outside diameter and 20 mm inside diameter is subjected to a torque of 50 N-m. Find the shear stress induced at the outside and inside layer of shaft.

Given : Outside diameter, $d_1 = 25$ mm

Inside diameter, $d_2 = 20$ mm

Torque transmitted, $T = 50$ N-m $= 50 \times 10^3$ N-mm

To find : 1) Shear stress at outside layer, f_{s1}

2) Shear stress at inside layer, f_{s2}

Solution :

Polar moment of inertia, $J = \frac{\pi}{32}[d_1^4 - d_2^4] = \frac{\pi}{32}[25^4 - 20^4] = 22641.556$ mm^4

We know that, $\frac{T}{J} = \frac{f_s}{r} \Longrightarrow f_s = \frac{T}{J} \times r$

At the outside layer, $r = r_1 = \frac{d_1}{2} = \frac{25}{2} = 12.5$ mm

$$f_{s1} = \frac{T}{J} \times r_1 = \frac{50 \times 10^3}{22641.556} \times 12.5 = \boxed{27.6 \text{ N/mm}^2}$$

At the inside layer, $r = r_2 = \frac{d_2}{2} = \frac{20}{2} = 10$ mm

$$f_{s2} = \frac{T}{J} \times r_2 = \frac{50 \times 10^3}{22641.556} \times 10 = \boxed{22.08 \text{ N/mm}^2}$$

Result : 1) Shear stress at outside layer, $f_{s1} = 27.6$ **N/mm^2**

2) Shear stress at inside layer, $f_{s2} = 22.08$ **N/mm^2**

Example : 7.8

A hollow shaft is to transmit 200KW at 80 rpm. If the stress is not to exceed 60 N/mm^2 and internal diameter is 0.6 times of the external diameter, find the diameter of the shaft.

Given : Power transmitted, $P = 200$ KW $= 200 \times 10^6$ N-mm/s

Speed of the shaft, $N = 80$ rpm

Allowable shear stress, $f_s = 60$ N/mm^2

Internal diameter, $d_2 = 0.6 \times$ External diameter (d_1)

To find : 1) External diameter, d_1 2) Internal diameter, d_2

Solution :

Torque transmitted by the hollow circular shaft,

$$T = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1}$$

$$= \frac{\pi \times 60}{16} \times \frac{d_1^4 - (0.6 d_1)^4}{d_1} = 10.254\, d_1^3 \text{ N-mm}$$

Power transmitted by the shaft,

$$P = \frac{2\pi N T}{60} = \frac{2\pi \times 80 \times 10.254\, d_1^3}{60} = 85.904\, d_1^3$$

$$200 \times 10^6 = 85.904\, d_1^3$$

$$d_1^3 = \frac{200 \times 10^6}{85.904} = 2.328 \times 10^6$$

$d_1 =$ **132.5 mm** $d_2 = 0.6 \times d_1 = 0.6 \times 132.5 =$ **79.5 mm**

Result : 1) External diameter, d_1 = **132.5 mm**
2) The internal diameter, d_2 = **79.5 mm**

Example : 7.9

(Apr.93)

A solid circular shaft has to transmit a power of 40KW at 120rpm. The permissible shear stress is 100N/mm². Determine the diameter of the shaft, if the maximum torque exceeds the mean torque by 25%.

Given : Power transmitted, P = 40 KW

Shear stress, $f_s = 100\ N/mm^2$

Maximum torque, $T_{max} = 1.25 \times$ Mean torque $= 1.25\, T_{mean}$

To find : 1) Diameter of shaft, d

Solution :

Power transmitted by the shaft,

$$P = \frac{2\pi N T_{mean}}{60}$$

$$T_{mean} = \frac{P \times 60}{2 \times \pi \times N} = \frac{40 \times 60}{2 \times \pi \times 120} = 3.183\ KN\text{-}m = 3.183 \times 10^6\ N\text{-}mm$$

$$T_{max} = 1.25 \times T_{mean} = 1.25 \times 3.183 \times 10^6 = 3.979 \times 10^6\ N\text{-}mm$$

Torque transmitted by the shaft,

$$T_{max} = \frac{\pi}{16} f_s d^3$$

$$d^3 = \frac{16 \times T_{max}}{\pi \times f_s} = \frac{16 \times 3.979 \times 10^6}{\pi \times 100} = 202648.806$$

d = **58.737 mm**

Result : 1) Diameter of shaft, d = **58.737 mm**

Example : 7.10 *(Oct.91, Oct.96)*

Find the torque transmitted by (i) solid shaft of diameter 0.4m (ii) hollow shaft of external diameter 0.4m and internal diameter 0.2m, if the angle of twist is not to exceed 1° in a length of 10m. Take $C = 0.8 \times 10^5 N/mm^2$.

Given : Angle of twist, $\theta = 1° = 1 \times (\pi/180) = 0.01745$ rad.

Modulus of rigidity, $C = 0.8 \times 10^5 N/mm^2$

Length of the shaft, $l = 10$ m $= 10000$ mm

To find : 1) Torque transmitted, T

Solution :

(i) Solid shaft

Diameter of the shaft, d = 0.4 m = 400mm

Polar moment of inertia, $J = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times 400^4 = 25.133 \times 10^8 \ mm^4$

Relation for troque transmitted by the shaft, $\frac{T}{J} = \frac{C\theta}{l}$

$$T = \frac{C\theta}{l} \times J = \frac{0.8 \times 10^5 \times 0.01745 \times 25.133 \times 10^8}{10000}$$

$= 3.509 \times 10^8$ N-mm $= 3.509 \times 10^2$ KN-m = **350.9 KN-m**

(ii) Hollow shaft

External diameter of the shaft, $d_1 = 0.4$ m = 400mm

Internal diameter of the shaft, $d_2 = 0.2$ m = 200 mm

Polar moment of inertia, $J = \frac{\pi}{32}(d_1^4 - d_2^4) = \frac{\pi}{32}(400^4 - 200^4)$

$= 23.562 \times 10^8 \ mm^4$

Relation for troque transmitted by the shaft, $\frac{T}{J} = \frac{C\theta}{l}$

$$T = \frac{C\theta}{l} \times J = \frac{0.8 \times 10^5 \times 0.01745 \times 23.562 \times 10^8}{10000}$$

$= 3.289 \times 10^8$ N-mm $= 3.289 \times 10^2$ KN-m = **328.9 KN-m**

Result : 1) Torque transmitted by solid shaft, T = **350.9 KN–m**

2) Torque transmitted by hollow shaft, T = **328.9 KN–m**

Example : 7.11

Find the angle of twist per metre length of a hollow shaft of 100mm external diameter and 60mm internal diameter, if the shear stress is not to exceed 35N/mm². Take $C = 85 \times 10^3 N/mm^2$.

Given : Length of the shaft, l = 1m = 1000 mm

External diameter, d_1 = 100 mm

Internal diameter, d_2 = 60 mm

Maximum shear stress, f_s = 35 N/mm²

Modulus of rigidity, C = $85 \times 10^3 N/mm^2$

To find : 1) Angle of twist, &

Solution :

Torque transmitted by the hollow circular shaft,

$$T = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi}{16} \times 35 \times \frac{100^4 - 60^4}{100} = 5.9816 \times 10^6 \text{ N-mm}$$

Polar moment of inertia, $J = \frac{\pi}{32}(d_1^4 - d_2^4) = \frac{\pi}{32}(100^4 - 60^4)$

$$= 8.5453 \times 10^6 mm^4$$

Relation for angle of twist,

$$\frac{T}{J} = \frac{C\&}{l}$$

$$\& = \frac{T\,l}{C\,J} = \frac{5.9816 \times 10^6 \times 1000}{85 \times 10^3 \times 8.5453 \times 10^6}$$

$$= 8.235 \times 10^{-3} \text{ rad.} = 8.235 \times 10^{-3} \times \frac{180}{\pi} = 0.472°$$

Result : 1) Angle of twist in the shaft, & = 0.472°

Example : 7.12

A solid shaft of 120mm diameter is required to transmit 200KW at 100 rpm. If the angle of twist is not to exceed 2°, find the length of the shaft. Take $C = 90 \times 10^3 N/mm^2$.

Given : Diameter of the shaft, d = 120 mm

Power transmitted, P = 200 KW

Speed of the shaft, N = 100 rpm

Angle of twist, & = 2° = 2 × (π /180) = 0.0349 rad.

Modulus of rigidity, C = $90 \times 10^3 N/mm^2$

To find : 1) Length of shaft, l

Solution :

Power transmitted by the shaft, $P = \frac{2 \pi N T}{60}$

$$T = \frac{P \times 60}{2 \times \pi \times N} = \frac{200 \times 60}{2 \times \pi \times 100} = 19.1 \text{ KN-m} = 19.1 \times 10^6 \text{ N-mm}$$

Polar moment of inertia, $J = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times 120^4 = 20.358 \times 10^6 \text{mm}^4$

Relation for length of the shaft,

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$l = \frac{C\theta \times J}{T} = \frac{90 \times 10^3 \times 0.0349 \times 20.358 \times 10^6}{19.1 \times 10^6} = 3347.878 \text{ mm}$$

Result : 1) Length of shaft, l = **3347.878 mm = 3.348 m**

Example : 7.13

(Oct.04, Oct.13, Oct.18)

A solid shaft 20mm diameter transmits 10KW at 1200 rpm. Calculate the maximum intensity of shear stress induced and the angle of twist in degrees in a length of 1m, if modulus of rigidity for the material of the shaft is $8 \times 10^4 \text{N/mm}^2$.

Given :

Diameter of the shaft, d = 20 mm

Power transmitted, P = 10 KW

Speed of the shaft, N = 1200 rpm

Length of the shaft, l = 1 m = 1000 mm

Modulus of rigidity, C = $8 \times 10^4 \text{N/mm}^2$

To find : 1) Shear stress, f_s 2) Angle of twist, θ

Solution :

Power transmitted by the shaft,

$$P = \frac{2 \pi N T}{60}$$

$$T = \frac{P \times 60}{2 \times \pi \times N} = \frac{10 \times 60}{2 \times \pi \times 1200}$$

$$= 79.577 \times 10^{-3} \text{ KN-m} = 79.577 \times 10^3 \text{ N-mm}$$

Torque transmitted by the shaft, $T = \frac{\pi}{16} f_s d^3$

$$f_s = \frac{16 \times T}{\pi \times d^3} = \frac{16 \times 79.577 \times 10^3}{\pi \times 20^3} = 50.66 \text{ N/mm}^2$$

Polar moment of inertia, $J = \frac{л}{32} d^4 = \frac{л}{32} \times 20^4 = 15.708 \times 10^3 mm^4$

Relation for angle of twist $\Longrightarrow \frac{T}{J} = \frac{C\&}{l}$

$\& = \frac{T\,l}{C\,J} = \frac{79.577 \times 10^3 \times 1000}{8 \times 10^4 \times 15.708 \times 10^3}$

$= 0.0633$ rad. $= 0.0633 \times \frac{180}{л} =$ **3.628°**

Result : 1) Shear stress induced, f_s = **50.66** N/mm^2
2) Angle of twist, & = **3.628°**

Example : 7.14

(Apr.04)

Calculate the power transmitted by a shaft of diameter 150mm at 120 rpm, if the maximum shear stress is not to exceed 80 N/mm^2***. What will be the angle of twist in a length of 10m? Take*** $C = 0.84 \times 10^5 N/mm^2$.

Given : Diameter of the shaft, d = 150 mm

Speed of the shaft, N = 120 rpm

Maximum shear stress, $f_s = 80\ N/mm^2$

Length of the shaft, l = 10 m = 10000 mm

Modulus of rigidity, $C = 0.84 \times 10^5 N/mm^2$

To find : 1) Power transmitted, P 2) Angle of twist, &

Solution :

Torque transmitted by the shaft,

$T = \frac{л}{16} f_s d^3 = \frac{л}{16} \times 80 \times 150^3 = 53.014 \times 10^6$ N-mm = 53.014 KN-m

Power transmitted by the shaft,

$P = \frac{2\,л\,N\,T}{60} = \frac{2 \times л \times 120 \times 53.014}{60} =$ **666.194 KW**

Polar moment of inertia, $J = \frac{л}{32} d^4 = \frac{л}{32} \times 150^4 = 49.7 \times 10^6 mm^4$

Relation for angle of twist $\Longrightarrow \frac{T}{J} = \frac{C\&}{l}$

$\& = \frac{T\,l}{C\,J} = \frac{53.014 \times 10^6 \times 10000}{0.84 \times 10^5 \times 49.7 \times 10^6}$

$= 0.127$ rad. $= 0.127 \times \frac{180}{л} =$ **7.276°**

Result : 1) Power transmitted, P = **666.194 KW**
2) Angle of twist, & = 7.276°

Example : 7.15 *(Apr.04)*

Find the maximum torque that can be applied to a shaft of 80mm diameter. The permissible angle of twist is 1.5° in a length of 5m and shear stress not to exceed 42 N/mm^2. ***Take*** $C = 84 \times 10^3 N/mm^2$.

Given : Diameter of shaft, d = 80 mm

Angle of twist, & = 1.5° = 1.5 × (π /180) = 0.02618 rad.

Length of the shaft, l = 5 m = 5000 mm

Maximum shear stress, $f_s = 42\ N/mm^2$

Modulus of rigidity, $C = 84 \times 10^3\ N/mm^2$

To find : 1) Torque that can be applied, T

Solution :

(a) Torque based on shear stress.

$$T_1 = \frac{\pi}{16} f_s d^3 = \frac{\pi}{16} \times 42 \times 80^3 = 4.222 \times 10^6 \text{ N-mm}$$

(b) Torque based on angle of twist

Polar moment of inertia, $J = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times 80^4 = 4.021 \times 10^6 mm^4$

Relation for torque $\Rightarrow \frac{T_2}{J} = \frac{C\&}{l}$

$$T_2 = \frac{C \& \times J}{l} = \frac{84 \times 10^3 \times 0.02618 \times 4.021 \times 10^6}{5000}$$

$$= 1.769 \times 10^6 \text{ N-mm}$$

We shall apply the torque which is lesser.

i.e T = T_2 = 1.769×10^6 N-mm

Result : 1) Torque that can be applied, T = 1.769×10^6 N-mm

Example : 7.16 *(Oct.89)*

The external and internal diameters of a hollow shaft are 400mm and 200mm respectively. Find the maximum torque that can be transmitted, if the angle of twist is not to exceed 0.5° in a length of 10m and the shear stress is not to exceed 70 N/mm^2. ***Take*** $C = 80\ KN/mm^2$.

Given : External diameter, $d_1 = 400$ mm

Internal diameter, $d_2 = 200$ mm

Angle of twist, & = 0.5° = 0.5 × (π /180) = 8.727×10^{-3} rad.

Length of the shaft, l = 10 m = 10000 mm

Maximum shear stress, $f_s = 70\ N/mm^2$

Modulus of rigidity, $C = 80\ KN/mm^2 = 80 \times 10^3 N/mm^2$

To find : 1) Maximum torque that can be transmitted, T

Solution :

(a) Torque based on shear stress

$$T_1 = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi}{16} \times 70 \times \frac{400^4 - 200^4}{400}$$

$= \boxed{8.247 \times 10^8 \text{ N-mm}}$

(b) Torque based on angle of twist

Polar moment of inertia, $J = \frac{\pi}{32}(d_1^4 - d_2^4) = \frac{\pi}{32}(400^4 - 200^4)$

$= 2.3562 \times 10^9 \text{ mm}^4$

Relation for torque $\Rightarrow \frac{T_2}{J} = \frac{C\theta}{l}$

$$T_2 = \frac{C\theta \times J}{l} = \frac{80 \times 10^3 \times 8.727 \times 10^{-3} \times 2.3562 \times 10^6}{10 \times 10^3}$$

$= \boxed{1.645 \times 10^8 \text{ N-mm}}$

We shall apply the torque which is lesser.

i.e. $T = T_2 = 1.645 \times 10^8$ N-mm

Result : 1) Torque that can be transmitted, $T = 1.645 \times 10^8$ **N-mm**

Example : 7.17 *(Oct.03)*

A solid shaft is subjected to a torque of 15KN–m. Find the suitable diameter of the shaft, if the allowable shear stress is 60 N/mm^2***. The allowable twist is 1° for every 20 diameters length of the shaft. Take*** C = 80 KN/mm^2.

Given : Torque, T = 15 KN–m = 15×10^6 N–mm

Angle of twist, $\theta = 1° = 1 \times (\pi/180) = 0.1745$ rad.

Length of the shaft, l = 20 × diameter (d)

Maximum shear stress, $f_s = 60$ N/mm^2

Modulus of rigidity, C = 80 KN/mm^2 = 80×10^3 N/mm^2

To find : 1) Diameter of shaft, d

Solution :

(a) Diameter for strength

Torque transmitted, $T = \frac{\pi}{16} f_s d^3$

$$d^3 = \frac{16 \times T}{\pi \times f_s} = \frac{16 \times 15 \times 10^6}{\pi \times 60} = 1.27324 \times 10^6$$

$d = \boxed{108.385 \text{ mm}}$

(b) Diameter for stiffness

Polar moment of inertia, $J = \frac{\pi}{32} d^4 = 0.098175\, d^4$

Relation for diameter $\Longrightarrow \frac{T}{J} = \frac{C\theta}{l}$

$$\frac{15 \times 10^6}{0.098175\, d^4} = \frac{80 \times 10^3 \times 0.01745}{20 \times d}$$

$$\frac{152.788 \times 10^6}{d^4} = \frac{69.8}{d}$$

$$d^3 = \frac{152.796 \times 10^6}{69.8} = 2.1889 \times 10^6$$

$$d = \boxed{129.84 \text{ mm}}$$

We shall provide a shaft of greater diameter.

i.e. d = 129.84 mm

Result : 1) Diameter of shaft, d = **129.84 mm**

Example : 7.18

(Apr.01, Apr.15, Apr.17)

A solid shaft is transmitting 100 KW at 180 rpm. If the allowable stress is 60 N/mm^2, find the necessary diameter for the shaft. The shaft is not to twist more than 1° in a length of 3 m. Take C = 80 KN/mm^2.

Given :

Speed of the shaft, N = 180 rpm

Power transmitted, P = 100 KW

Maximum shear stress, $f_s = 60\ N/mm^2$

Angle of twist, $\theta = 1° = 1 \times (\pi/180) = 0.01745$ rad.

Length of the shaft, l = 3 m = 3000 mm

Modulus of rigidity, C = 80 KN/mm^2 = $80 \times 10^3\ N/mm^2$

To find : 1) Diameter of shaft, d

Solution :

Power transmitted by the shaft, $P = \frac{2 \pi N T}{60}$

$$T = \frac{P \times 60}{2 \pi N} = \frac{100 \times 60}{2 \pi \times 180} = 5.3052 \text{ KN-m} = 5.3052 \times 10^6 \text{ N-mm}$$

(a) Diameter for strength

Torque transmitted, $T = \frac{\pi}{16} f_s\, d^3$

$$d^3 = \frac{16 \times T}{\pi \times f_s} = \frac{16 \times 5.3052 \times 10^6}{\pi \times 60} = 450319.36$$

$$d = 76.65 \text{ mm} \approx \boxed{77 \text{ mm}}$$

(b) Diameter for stiffness

Polar moment of inertia, $J = \frac{\pi}{32} d^4$

Relation for diameter $\Longrightarrow \frac{T}{J} = \frac{C\theta}{l}$

$\frac{T \times 32}{\pi d^4} = \frac{C\theta}{l}$

$d^4 = \frac{T \times 32 \times l}{\pi \times C \times \theta} = \frac{5.3052 \times 10^6 \times 32 \times 3000}{\pi \times 80 \times 10^3 \times 0.01745} = 116.128 \times 10^6$

$d = 103.809$ **mm** ≈ **104 mm**

We shall provide a shaft of greater diameter.

i.e., d = 104 mm

Result : 1) Diameter of shaft, d = **109.76 mm**

Example : 7.19

A solid steel shaft of 60mm diameter is to be replaced by a hollow steel shaft of the same material with internal diameter equal to half of the external diameter. Find the diameters of the hollow shaft and saving in material, if the maximum allowable shear stress is same for both the shafts.

Given : Diameter of solid shaft, d = 60 mm

External diameter of hollow shaft, $d_1 = 0.5 \times$ Internal diameter (d_2)

To find : 1) Diameters of the hollow shaft, d_1 and d_2

2) Percent saving in material

Solution :

Torque transmitted by the solid shaft,

$T_1 = \frac{\pi}{16} f_s d^3 = \frac{\pi}{16} \times f_s \times 60^3$ --------- (1)

Torque transmitted by the hollow shaft,

$T_2 = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi}{16} \times f_s \times \frac{d_1^4 - (0.5 d_1)^4}{d_1}$

$T_2 = \frac{\pi}{16} \times f_s \times 0.9375\, d_1^3$ --------- (2)

Power transmitted and allowable shear stress in both the cases are same

$\therefore T_1 = T_2$

$\frac{\pi}{16} \times f_s \times 60^3 = \frac{\pi}{16} \times f_s \times 0.9375\, d_1^3$

$d_1^3 = \frac{60^3}{0.9375} = 230400$

$d_1 =$ **61.305 mm** ; $d_2 = \frac{d_1}{2} = \frac{61.305}{2} =$ **30.653 mm**

Area of the solid shaft, $A_s = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 60^2 = 2827.433\ mm^2$

Area of the hollow shaft, $A_h = \frac{\pi}{4} \times (d_1^2 - d_2^2) = \frac{\pi}{4} \times (61.305^2 - 30.653^2) = 2213.799\ mm^2$

Saving in material, $= \frac{A_s - A_h}{A_s} \times 100 = \frac{2827.433 - 2213.799}{2827.433} \times 100 =$ **21.7 %**

Result : 1) External diameter of hollow shaft, d_1 = **61.305 mm**
2) Internal diameter of hollow shaft, d_2 = **30.635 mm**
3) Saving in material = **21.7 %**

Example : 7.20 *(Apr.13, Apr.14, Oct.16)*

A hollow shaft having inner diameter 0.6 times the outer diameter is to be replaced by a solid shaft of the same material to transmit 550KW at 220 rpm. The permissible shear stress is 80 N/mm^2***. Calculate the diameters of the hollow and solid shafts. Also calculate the percentage of saving in material.***

Given : Power transmitted, P = 550 KW
Speed of the shaft, N = 220 rpm
Shear stress, $f_s = 80\ N/mm^2$

To find : 1) Diameter of solid shaft, d
2) Diameters of hollow shaft, d_1 and d_2
3) Percentage saving in material

Solution :

Power transmitted by the shaft, $P = \frac{2 \pi N T}{60}$

$T = \frac{P \times 60}{2 \pi N} = \frac{550 \times 60}{2 \pi \times 220} = 23.873\ KN\text{-}m = 23.873 \times 10^6\ N\text{-}mm$

(a) Solid shaft

Torque transmitted, $T = \frac{\pi}{16} f_s d^3$

$d^3 = \frac{16 \times T}{\pi \times f_s} = \frac{16 \times 23.873 \times 10^6}{\pi \times 80} = 1519802.383$

d = **114.973 mm**

(b) Hollow shaft

Torque transmitted by the hollow shaft,

$$T = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi \times 80}{16} \times \frac{d_1^4 - (0.6 d_1)^4}{d_1}$$

$23.873 \times 10^6 = 13.672\, d_1^3$

$23.873 \times 10^6 = 13.672\, d_1^3$

$$d_1^3 = \frac{23.873 \times 10^6}{13.67} = 1746123.464$$

$d_1 =$ **120.418 mm**

$d_2 = 0.6\, d_1 = 0.6 \times 120.418 =$ **72.251 mm**

Area of the solid shaft, $A_s = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 114.973^2 = 10382\ mm^2$

Area of the hollow shaft, $A_h = \frac{\pi}{4} \times (d_1^2 - d_2^2) = \frac{\pi}{4} \times (120.418^2 - 72.251^2) = 7288.72\ mm^2$

Saving in material, $= \frac{A_s - A_h}{A_s} \times 100 = \frac{10382 - 7288.72}{10382} \times 100 =$ **29.79 %**

Result : 1) Diameter of solid shaft, d = **114.973 mm**

2) External diameter of hollow shaft, d_1 = **120.418 mm**

3) Internal diameter of hollow shaft, d_2 = **72.251 mm**

4) Saving in material = **29.79 %**

Example : 7.21

(Oct.92)

Compare the weight of a solid shaft with that of a hollow shaft for the same material, length and designed to reach the same maximum shear stress when subjected to same torque. Assume the inside diameter of the hollow shaft equal to two third of the external diameter.

Solution :

Let, T = Torque transmitted by the shaft, f_s = Maximum shear stress

l = Length of the shaft

(a) Solid shaft

Let, d = Diameter of solid shaft

Torque transmitted by the shaft, $T = \frac{\pi}{16} f_s d^3$

$$d^3 = \frac{16 \times T}{\pi \times f_s} = 5.093 \left(\frac{T}{f_s}\right)$$

$$d = 1.7205 \left(\frac{T}{f_s}\right)^{\frac{1}{3}}$$

Weight of the solid shaft,

$$W_1 = plA_1 = pl \times \frac{\pi}{4} d^2 = pl \times \frac{\pi}{4} \left[1.7205 \left(\frac{T}{f_s}\right)^{\frac{1}{3}}\right]^2$$

$$= 2.3249\ pl \left(\frac{T}{f_s}\right)^{\frac{2}{3}}$$

(b) Hollow shaft

Let, d_1 = External diameter, d_2 = Internal diameter

Then, $d_2 = \frac{2}{3} d_1 = 0.667\ d_1$

Torque transmitted by the hollow shaft,

$$T = \frac{\pi}{16} \times f_s \times \frac{(d_1^4 - d_2^4)}{d_1} = \frac{\pi \times f_s}{16} \times \frac{d_1^4 - (0.667 d_1)^4}{d_1}$$

$$T = 0.157488\ f_s\ d_1^3$$

$$d_1^3 = \frac{T}{0.157488\ f_s} = 6.3497 \frac{T}{f_s}$$

$$d_1 = 1.8518 \left(\frac{T}{f_s}\right)^{\frac{1}{3}}$$

$$d_2 = 0.667 \times d_1 = 0.667 \times 1.8518 \left(\frac{T}{f_s}\right)^{\frac{1}{3}} = 1.235 \left(\frac{T}{f_s}\right)^{\frac{1}{3}}$$

Weight of the hollow shaft,

$$W_2 = plA = pl \times \frac{\pi}{4} (d_1^2 - d_2^2)$$

$$= pl \times \frac{\pi}{4} \left\{ \left[1.8518 \left(\frac{T}{f_s}\right)^{\frac{1}{3}}\right]^2 - \left[1.235 \left(\frac{T}{f_s}\right)^{\frac{1}{3}}\right]^2 \right\}$$

$$= 1.4954\ pl \left(\frac{T}{f_s}\right)^{\frac{2}{3}}$$

The ratio of weight of solid shaft to hollow shaft,

$$\frac{W_1}{W_2} = \frac{2.3249\ pl \left(\frac{T}{f_s}\right)^{\frac{2}{3}}}{1.4954\ pl \left(\frac{T}{f_s}\right)^{\frac{2}{3}}} = 1.5547$$

Result : 1) The ratio of weight of solid shaft to hollow shaft = **1.5547**

Unit – IV

Chapter 8.
SPRINGS

1. Introduction

A spring is a device which can undergo considerable amount of deformation without permanent distortion. The general purpose of all kinds of springs is to absorb energy and to release it as and when required. Springs are also used to provide a means of restoring various mechanisms to their original configurations against the action of some external force.

1. Types of springs

The springs are classified as follows based on their forms :

1) Laminated or leaf springs
2) Coiled helical springs
3) Spiral springs
4) Disc springs

1) Laminated or leaf springs

Fig.8.1 Laminated or Leaf spring

A laminated spring consists of a number of arc shaped strips of metal having different lengths but same width and thickness. They are placed one over the other in laminations. The strips are bolted together. The two types of laminated springs are :

(i) Semi - elliptical laminated springs
(ii) Quarter - elliptical laminated springs.

Uses : Thsese springs are used in railway wagons, coaches and road vehicles to absorb shocks.

2) Coiled helical springs

A helical spring is made up of a wire wound in helix form. The following two types of helical springs are used.

i) Closely coiled helical spring ii) Open coiled helical spring

Comparison of closely coiled helical spring and open coiled helical spring

	Closely coiled helical spring	Open coiled helical spring
1)	The pitch of the coil is very small	The pitch of the coil is large
2)	The gap between the successive turn is small	The gap between the successive turn is large
3)	The helix angle is less (7° to 10°)	The helix angle is more ($>10^{\circ}$C)
4)	Under axial load, it is subjected to torsion only	It is subjected to both torsion and bending
5)	It can withstand more load	It can withstand less load

The helical springs are further classified as

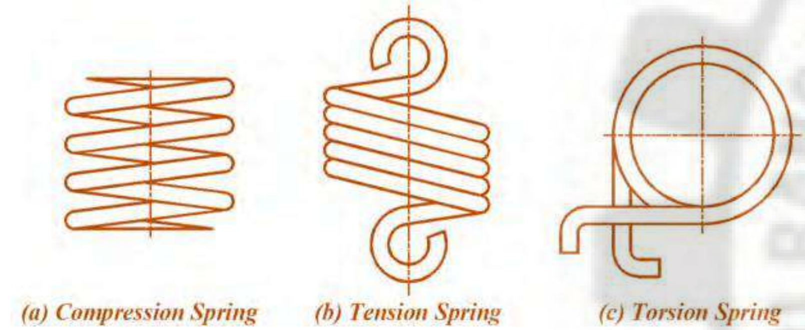


Fig.8.2 Coiled helical springs

(a) Compression springs

A helical spring is said to be a compression spring, if the coils close when subjected to axial load and open out when the load is removed.

Uses : These springs are used in automobiles and railway coaches as shock absorbers.

(b) Tension springs

A helical spring is said to be a tension spring, if the coils open out when subjected to axial load and closes when the load is removed.

Uses : These springs are used in spring balances and cycle stands.

(c) Torsion springs or extension springs

The coils of torsion springs are fully compressed. Both the ends of the coil are straightened out. When one end is fixed and other end rotated, the coil deforms and creates a force opposing the rotation.

Uses : These springs are used in mouse trap, automobile starters, door hinges, etc.

3) Spiral springs or constant force springs

It consists of a uniform thin strip wound into a spiral shape. The outer end is pinned. The inner end is wound on a spindle by applying a torque. The

wound spring is released slowly over a period of time. It gives a

Fig.8.3 Spiral spring

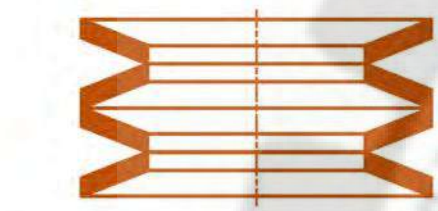


Fig.8.4 Disc spring

4) Disc springs or Belleville washer

It is a convex disc shaped spring with a hole at the centre. It can be used singly or in stacks to achieve a desired load. This spring requires less space for installation. It can withstand a very large load.

Uses : These springs are used in clutches, high pressure valves, drill bit shock absorbers, etc.

8.3 Closely coiled helical spring subjected to an axial load

Consider a closely coiled helical spring subjected to an axial load as shown in the fig.8.5.

Let, d = Diameter

of the spring wire R = Mean

radius of the spring coil н

= Number of turn

C = Modulus of rigidity of spring material

W = Axial load the spring

f_s = Maximum shear stress induced in the wire due to twisting

& = Angle of twist in the spring wire and

ð = Deflection of the spring due to axial load

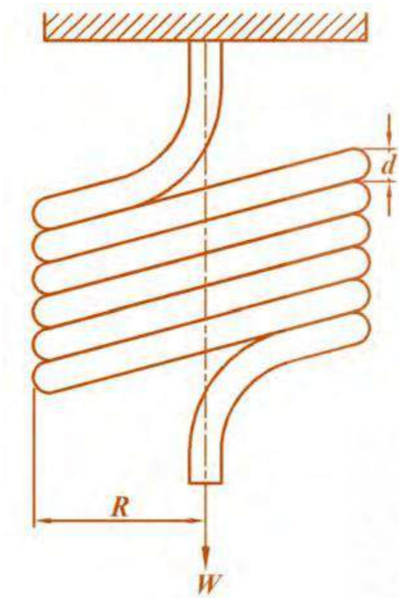


Fig.8.5 Closely coiled helical spring

Twisting moment on the coil due to the axial load, T = W . R – – – –(1)

We know that, $T = \frac{л}{16} f_s d^3$ – – – – (2)

$$\therefore WR = \frac{л}{16} f_s d^3$$

$$f_s = \frac{16\, W R}{л d^3}$$

Length of the wire, l = 2 л R . н

From the equation, $\frac{T}{J} = \frac{C\&}{l}$

$$\& = \frac{T\, l}{C\, J} = \frac{WR \times 2л R н}{C \times \frac{л}{32} d^4}$$

$$\& = \frac{64\, W R^2 н}{C\, d^4}$$

Deflection of the spring, $ð = R\& = R \times \frac{64\, W R^2 н}{C\, d^4}$

$$6 = \frac{64\, W R^3 н}{C\, d^4}$$

8.4 Stiffness of the spring

The stiffness of the spring is defined as the load required to produce unit deflection. It is denoted by 's'.

$$s = \frac{W}{\delta} = \frac{W}{\frac{64 W R^3 H}{C d^4}} = \frac{C d^4}{64 R^3 H}$$

It is also known as ***spring constant.***

8.5 Resilience or strain energy stored in a closely coiled helical spring.

Energy stored = Average load × Deflection

$$= \frac{W}{2} \times \frac{64 W R^3 H}{C d^4} = \frac{32 W^2 R^3 H}{C d^4}$$

8.6 Applications of springs

1) To apply forces and controlling motion, as in brakes and clutches.
2) Measuring forces, as in spring balances.
3) Storing energy, as springs used in watches and toys.
4) Reducing the effect of shock and vibrations in vehicles and machine foundations.

SOLVED PROBLEMS

Example : 8.1 *(Apr.89, Oct.90)*

A closely coiled helical spring of alloy steel wire of 10mm diameter having 15 complete turns with the mean coil diameter as 10mm. Calculate the stiffness of the spring. Take $C = 90 \times 10^3\ N/mm^2$.

Given :

Diameter of wire, d = 10 mm

Mean diameter of coil, D = 100 mm

Number of turns, n = 15

Modulus of rigidity, $C = 90 \times 10^3\ N/mm^2$

To find : 1) Stiffness of spring, s

Solution :

Mean radius, $R = \frac{D}{2} = \frac{100}{2} = 50$ mm

The stiffness of spring, $s = \frac{Cd^4}{64 R^3 n} = \frac{90 \times 10^3 \times 10^4}{64 \times 50^3 \times 15} = $ **7.5 N/mm**

Result : 1) Stiffness of spring, s = **7.5 N/mm**

Example : 8.2 *(Oct.03)*

Calculate the modulus of rigidity of a spring of 10 turns 65mm mean diameter and wire of 6.5mm diameter. The spring compresses 10mm under a load of 70N.

Given :

Number of turns, n = 10

Mean diameter of coil, D = 65 mm

Diameter of wire, d = 6.5 mm

Load, W = 70 N

Deflection, δ = 10 mm

To find : 1) Modulus of rigidity, C

Solution :

Mean radius, $R = \frac{D}{2} = \frac{65}{2} = 32.5$ mm

Relation for modulus of rigidity $\Rightarrow \delta = \frac{64 WR^3 n}{Cd^4}$

$$C = \frac{64 WR^3 n}{\delta d^4} = \frac{64 \times 70 \times 32.5^3 \times 10}{10 \times 6.5^4} = 86.154 \times 10^3\ N/mm^2$$

Result : 1) Modulus of rigidity, C = $86.154 \times 10^3\ N/mm^2$

Example : 8.3 (Oct.92)

A closely coiled helical spring has the stiffness of 40N/mm. Determine its number of turns when the diameter of the wire of the spring is 10mm and mean diameter of the coil is 80mm. Take $C = 0.8 \times 10^5\ N/mm^2$.

Given : Stiffness, s = 40 N/mm

Mean diameter of coil, D = 80 mm

Diameter of wire, d = 10 mm

Modulus of rigidity, $C = 0.8 \times 10^5\ N/mm^2$

To find : 1) Number of turns in the spring, н

Solution :

$$\text{Mean radius, } R = \frac{D}{2} = \frac{80}{2} = 40 \text{ mm}$$

$$\text{Stiffness, } s = \frac{Cd^4}{64\, R^3 н}$$

$$н = \frac{Cd^4}{64\, R^3 s} = \frac{0.8 \times 10^5 \times 10^4}{64 \times 40^3 \times 40} = \boxed{5.2 \approx 6}$$

Result : 1) Number of turns in the spring, н = **6**

Example : 8.4 (Oct.15)

A closely coiled helical spring made of 12mm steel wire having 12 turns of mean radius 60mm elongates by 15mm under a load. Find the magnitude of the load if the modulus of rigidity is given as $7.5 \times 10^4\ N/mm^2$.

Given : Diameter of wire, d = 12 mm

Number fo turns, н = 12

Mean radius of coil, R = 60 mm

Deflection of spring, ð = 15 mm

Modulus of rigidity, $C = 7.5 \times 10^4\ N/mm^2$

To find : 1) Magnitude of load, W

Solution :

$$\text{Deflection of spring, } ð = \frac{64\, WR^3 н}{Cd^4}$$

$$W = \frac{ð \times C\, d^4}{64\, R^3\, н} = \frac{15 \times 7.5 \times 10^4 \times 12^4}{64 \times 60^3 \times 12} = \boxed{140.63\ \text{N}}$$

Result : 1) Magnitude of load, W = **140.63 N**

Example : 8.5

(Apr.01, Oct.13)

A closely coiled helical spring is to carry a load of 100KN. The mean coil diameter is 15 times that of the wire diameter. Calculate these diameters if the shear stress is limited to 120 N/mm^2.

Given : Load, W = 100 KN = 100×10^3 N

Shear stress, f_s = 120 N/mm^2

To find : 1) Diameter of wire, d 2) Diameter of coil, D

Solution :

Let, d = Diameter of wire ; D = Diameter of coil

Then, D = 15 × d ; $R = \frac{D}{2} = \frac{15\,d}{2} = 7.5\,d$

Torque, $T = W \times R = 100 \times 10^3 \times 7.5\,d = 7.5 \times 10^5\,d$

Also, torque, $T = \frac{\pi}{16} f_s d^3 = \frac{\pi}{16} \times 120 \times d^3 = 23.562\,d^3$

$\therefore 23.562\,d^3 = 7.5 \times 10^5\,d$

$d^2 = \frac{7.5 \times 10^5}{23.562} = 31830.91$

d = **178. 4 mm** ; D = 15 d = 15 × 178.4 = **2676 mm**

Result : 1) Diameter of wire, d **= 178.4 mm**
2) Diameter of coil, D **= 2676 mm**

Example : 8.6

(Apr.04, Oct.14, Apr.18)

The mean diameter of a closely coiled helical spring is 5 times the diameter of wire. It elongates 8mm under an axial pull of 120N. If the permissible shear stress is 40 N/mm^2, find the size of wire and number of coils in the spring. Take $C = 0.8 \times 10^5\ N/mm^2$.

Given : Deflection, ð = 8 mm

Axial load, W = 120 N

Shear stress, f_s = 40 N/mm^2

Modulus of rigidity, C = $0.8 \times 10^5\ N/mm^2$

To find : 1) Diameter of wire, d 2) Number of turns, n

Solution :

Let, d = Diameter of wire ; D = Diameter of coil

Then, D = 5 × d ; $R = \frac{D}{2} = \frac{5\,d}{2} = 2.5\,d$

Torque, T = W × R = 120 × 2.5 d = 300 d

Also, torque, $T = \frac{л}{16} f_s d^3 = \frac{л}{16} \times 40 \times d^3 = 7.854\, d^3$

$\therefore 7.854\, d^3 = 300\, d$

$d^2 = \frac{300}{7.854} = 38.197$

$d =$ **6.18 mm** ; $R = 2.5\, d = 2.5 \times 6.18 =$ **15.45 mm**

Relation for number of turns ⇒ $ð = \frac{64\, WR^3 н}{Cd^4}$

$н = \frac{Cd^4 \times ð}{64\, W\, R^3} = \frac{0.8 \times 10^5 \times 6.18^4 \times 8}{64 \times 120 \times 40^3} = 32.96 \approx$ **33**

Result : 1) Diameter of wire, d **= 6.18 mm**
2) Number of turns, н **= 33**

Example : 8.7

(Oct.02, Apr.14, Oct.16, Apr.17)

A closely coiled helical spring made of steel wire of 10mm diameter has 10 coils of 120mm mean diameter. Calculate the deflection of the spring under an axial load of 100N and the stiffness of the spring. Take $C = 1.2 \times 10^5\ N/mm^2$.

Given : Diameter of wire, d = 10 mm

Number of turns, н = 10

Mean diameter of coil, D = 120 mm

Axial load, W = 100 N

Modulus of rigidity, $C = 1.2 \times 10^5\ N/mm^2$

To find : 1) Deflection, ð 2) Stiffness, s

Solution :

Mean radius, $R = \frac{D}{2} = \frac{120}{2} = 60$ mm

Deflection, $ð = \frac{64\, WR^3 н}{Cd^4} = \frac{64 \times 100 \times 60^3 \times 10}{1.2 \times 10^5 \times 10^4} =$ **11.52 mm**

Stiffness, $s = \frac{W}{\delta} = \frac{100}{11.52} =$ **8.68 N/mm**

Result : 1) Deflection, 6 **= 11.52 mm** 2) Stiffness, s = **8.68 N/mm**

Example : 8.8

Oct.88, Apr.92, Apr.01, Oct.12, Apr.13

Design a closely coiled helical spring of stiffness 20N/mm deflection. The maximum shear stress in the spring material is not to exceed 80 N/mm^2 ***under a load of 600N. The diameter of the coil is to be 10 times the diameter of the wire. Take*** $C = 85 \times 10^3\ N/mm^2$.

Given : Stiffness of the spring, s = 20 N/mm

Shear stress, $f_s = 80\ N/mm^2$

Axial load, W = 600 N

Modulus of rigidity, $C = 85 \times 10^3\ N/mm^2$

Solution :

Let, d = Diameter of wire ; D = Diameter of coil

Then, D = 10 d ; $R = \frac{D}{2} = \frac{10\ d}{2} = 5\ d$

Torque, T = W × R = 600 × 5 d = 3000 d

Also, torque, $T = \frac{\pi}{16} f_s d^3 = \frac{\pi}{16} \times 80 \times d^3 = 15.708\ d^3$

$\therefore 15.708\ d^3 = 3000\ d$

$d^2 = \frac{3000}{15.708} = 190.986$

d = 13.82 mm ≈ **14 mm**

D = 10 d = 10 × 14 = **140 mm** ; R = 5d = 5 × 14 = **70 mm**

Relation for number of turns ⇒ $s = \frac{Cd^4}{64\ R^3 n}$

$$n = \frac{Cd^4}{64\ R^3 s} = \frac{85 \times 10^3 \times 14^4}{64 \times 70^3 \times 20} = 7.44 \approx 8$$

Result : 1) Diameter of coil, D **= 140 mm**

2) Diameter of wire, d **= 14 mm**

3) Number of turns, n **= 8**

Example : 8.9

A closely coiled helical spring is to be designed to carry an axial load 2500N under a deflection of 70mm. The number of coil is to be limited to 10 and the coil diameter is 10 times the wire diameter. Calculate the diameter of the coil and shear stress produced in the spring. Take $C = 85 KN/mm^2$.

Given : Axial load, W = 2500 N

Deflection, δ = 70 mm

Number of coil, n = 10

Modulus of rigidity, $C = 85\ KN/mm^2 = 85 \times 10^3\ N/mm^2$

To find : 1) Diameter of coil, D 2) Shear stress, f_s

Solution :

Let, d = Diameter of wire ; D = Diameter of coil

Then, D = 10 d ; $R = \frac{D}{2} = \frac{10\,d}{2} = 5\,d$

Deflection, $\delta = \frac{64\,WR^3 n}{Cd^4} = \frac{64 \times 2500 \times (5d)^3 \times 10}{85 \times 10^5 \times d^4}$

$70 = \frac{2352.94}{d}$

$d = \frac{2352.94}{70} = 33.61$ **mm** ≈ **34 mm**

D = 10 d = 10 × 34 = **340 mm**

Torque, T = W × R = 2500 × (5 × 34) = 425000 N-mm

Also, torque, $T = \frac{\pi}{16} f_s d^3$

$f_s = \frac{16\,T}{\pi\,d^3} = \frac{16 \times 425000}{\pi \times 34^3} = 55.07\ N/mm^2$

Result : 1) Diameter of coil, D = **340 mm** 2) Shear stress, f_s = **55.07** N/mm^2

Example : 8.10

(Oct.92)

A closely coiled helical spring has to absorb 50N–m of energy when compressed by 50mm. The coil diameter is 12 times the wire diameter. The number of coil is 10. Determine the diameters of the wire and coil, if $C = 0.08 \times 10^6 N/mm^2$.

Given : Energy absorbed = 50 N–m = 50×10^3 N–mm Deflection, δ = 50 mm

Number of coil, n = 10

Modulus of rigidity, $C = 0.08 \times 10^6\ N/mm^2$

To find : 1) Diameter of coil, D 2) Diameter of wire, d

Solution :

Let, d = Diameter of wire ; D = Diameter of coil

Then, D = 12 d ; $R = \frac{D}{2} = \frac{12\,d}{2} = 6\,d$

Energy absorbed by the coil = Average load × deflection

$50 \times 10^3 = \frac{W}{2} \times 50$

$W = \frac{2 \times 50 \times 10^3}{50} = 2000\ N$

$$\text{Deflection, } \delta = \frac{64\,WR^3 n}{Cd^4} = \frac{64 \times 2000 \times (6d)^3 \times 10}{0.08 \times 10^6 \times d^4}$$

$$50 = \frac{3456}{d}$$

$$d = \frac{3456}{50} = 69.12 \approx \boxed{70\text{ mm}}$$

$$D = 12\,d = 12 \times 70 = \boxed{840\text{ mm}}$$

Result : 1) Diameter of coil, D **= 840 mm** 2) Diameter of wire, d **= 70 mm**

Example : 8.11

(Oct.03, Oct.17)

A truck weighing 30KN and moving at 5Km/hr has to be brought to rest by a buffer. Find how many springs, each of 18 coils will be required to store the energy of motion during compression of 200mm. The spring is made out of 25mm diameter steel rod coiled to a mean diameter of 240mm. Take $C = 0.84 \times 10^5 N/mm^2$.

Given : Weight of the truck, $W_1 = 30\text{ KN} = 30 \times 10^3$ N

Velocity of the truck, $u = 5\text{Km/hr} = \frac{5 \times 10^3 \times 10^3}{60 \times 60} = 1388.889$ mm/s

Number of coil, $n = 18$

Deflection, $\delta = 200$ mm

Diameter of wire, d = 25 mm

Diameter of coil, D = 240 mm

Modulus of rigidity, $C = 0.84 \times 10^5\ N/mm^2$

To find : 1) Number of springs

Solution :

$$\text{Mean radius, } R = \frac{D}{2} = \frac{240}{2} = 120\text{ mm}$$

Kinetic energy stored in the truck,

$$K.E = \frac{W_1 u^2}{2g} = \frac{30 \times 10^3 \times 1388.889^2}{2 \times 9.81 \times 10^3} = 2.95 \times 10^6\text{ N-mm}$$

Let. W = Axial load act on each spring

$$\text{Then deflection, } \delta = \frac{64\,WR^3 n}{Cd^4}$$

$$W = \frac{Cd^4 \times \delta}{64\,R^3 n} = \frac{0.84 \times 10^5 \times 25^4 \times 200}{64 \times 120^3 \times 18} = 3296.65\text{ N}$$

Energy stored in each spring = Average load × deflection

$$= \frac{W}{2} \times \delta = \frac{3296.65}{2} \times 200 = 329665 \text{ N-mm}$$

$$\text{No. of springs} = \frac{\text{Kinetic energy stored in the truck}}{\text{Energy stored in each spring}}$$

$$= \frac{2.95 \times 10^6}{3296.65} = 8.95 \approx \boxed{9}$$

Result : 1) Number of springs required = 9

Example : 8.12 *(Oct.04, Oct.16)*

A weight of 150 N is dropped on to a compression spring with 10 coils of 12 mm diameter closely coiled to a mean diameter of 150 mm. If the instantaneous contraction is 140 mm, calculate the height of drop. Take $C = 0.8 \times 10^5 \text{N/mm}^2$.

Given : Weight dropped on the spring, P = 150 N

Number of turns, n = 10

Deflection, δ = 140 mm

Diameter of wire, d = 12 mm

Diameter of coil, D = 150 mm

Modulus of rigidity, $C = 0.8 \times 10^5 \text{ N/mm}^2$

To find : 1) Height of drop of weight, h

Solution :

$$\text{Mean radius, } R = \frac{D}{2} = \frac{150}{2} = 75 \text{ mm}$$

Let, h = Height of drop of weight before strike

Potential energy stored in the weight,

$$= P(h + \delta l) = 150\,(h + 140)$$

Let, W = Axial load on each spring

$$\text{Then, deflection, } \delta = \frac{64 W R^3 n}{C d^4}$$

$$W = \frac{Cd^4 \times \delta}{64 R^3 n} = \frac{0.8 \times 10^5 \times 12^4 \times 140}{64 \times 75^3 \times 10} = 860.16 \text{ N}$$

Energy stored in spring = Average load × deflection

$$= \frac{W}{2} \times \delta = \frac{860.16}{2} \times 140 = 60211.2 \text{ N-mm}$$

After striking,

the potential energy stored in the weight is lost to compress the spring.

∴ Potential energy stored in weight = Energy stored in spring

$$150(h + 140) = 60211.2$$

$$h + 140 = \frac{60211.2}{150} = 401.408 \text{ mm}$$

$$h = 401.408 - 140 = \boxed{261.408 \text{ mm}}$$

Result : 1) Height of drop of weight, h **= 261.408 mm**

Unit – V

Chapter 9. SHEAR FORCE AND BENDING MOMENT DIAGRAMS

1. Beam

Beam is a structural member which is subjected to a system of external forces acting perpendicular to its axis.

Whenever a beam is subjected to vertical loads it bends due to the action of the load. The amount with which a beam bends, depends upon the type of loads, length of the beam, elasticity of the beam and the type of beam.

1. Classification (

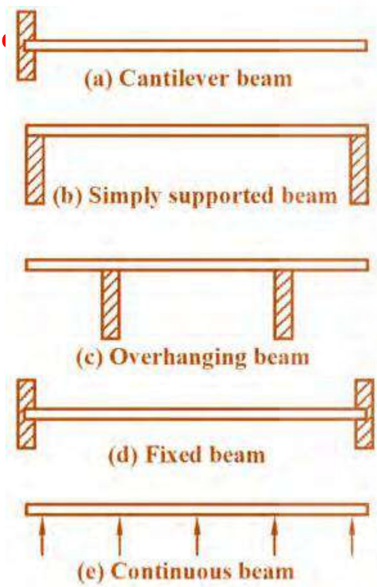


Fig.9.1 Types of beam

The beams are generally classified according to the supporting conditions as follows.

1) Cantilever beam
2) Simply supported beam
3) Overhanging beam
4) Fixed beam
5) Continuous beam

1) Cantilever beam

If one end of the beam is fixed and the other end is free, then such type of beam is called cantilever beam.

2) Simply supported beam

If both the ends of the beam are made to rest freely on supports, then such type of beam is called simply supported beam.

3) Overhanging beam

If the ends of the beam are extended beyond the supports in a simply supported beam, then it is called as overhanging beam.

4) Fixed beam

If both the ends of a beam are rigidly fixed or built into the walls, then it is called fixed beam.

5) Continuous beam

If a beam is provided with more than two supports, then it is called as continuous beam.

9.3 Types of loading

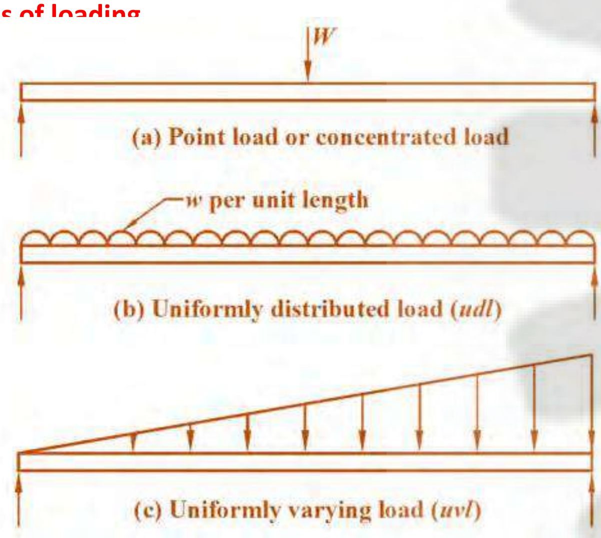


Fig.9.2 Types of loading

A beam may be subjected to the following types of loads.

1) Point load or concentrated load.
2) Uniformly distributed load (udl).
3) Uniformly varying load.

1) Point load or concentrated load

If a load is acting exactly at a point in the beam then it is called point load or concentrated load.

2) Uniformly distributed load (udl)

If a load is spread over the beam in such a way that its magnitude is same for each and every unit length of the beam, then it is called uniformly
distributed load (udl).

3) Uniformly varying load

If a load is spread over the beam in such a way that its magnitude is gradually varying within an unit length of the beam, then it is called uniformly varying load.

4. Shear force

The shear force at a cross section of beam may be defined as the unbalanced vertical forces to the left or right of the section. It is denoted as **SF.**

4. Bending moment

The bending moment at a cross section of a beam may be defined as the algebraic sum of the moments of the forces to the left or right of the
section. It is denoted as **BM.**

4. Sign conventions.

Shear forc

C ... the beam. All
the upwar ... ırd forces
to the rig

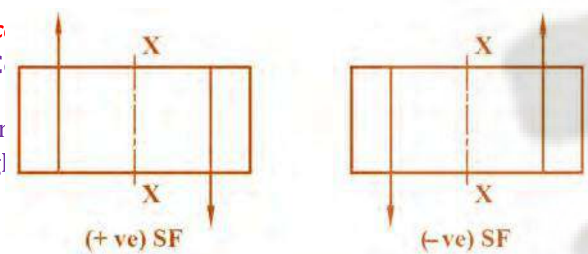


Fig.9.3 Sign convention of shear force

All the upward forces to the right of the section and all the downward forces to the left of the section cause negative shear force.

Bending moment

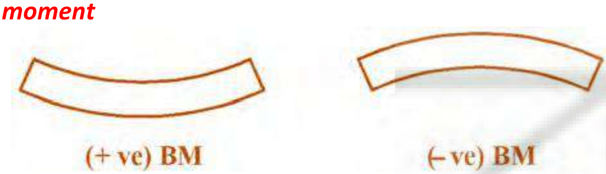


Fig.9.4 Sign convention of bending moment

If the bending moment at a section is such a way that it tends to bend the beam at that point to a curvature having concavity at the top is taken as positive bending moment. The positive bending moment is often called as ***sagging*** moment. The right anti–clockwise moment and left clockwise moment are taken as positive moment.

If the bending moment at a section is such a way that it tends to bend the beam at that point to a curvature having convexity at the top is taken as negative bending moment. The negative bending moment is often called as ***hogging*** moment. The right clockwise moment and left anti–clockwise moment are taken as negative moment.

9.7 Relationship between load, shear force and bending moment

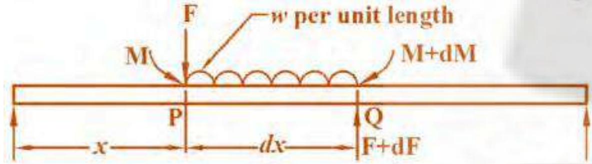


Fig.9.5 Relationship between load, SF and BM.

Consider a beam carrying a udl of r per unit length. Let us consider a portion PQ of length dz and at a distance z from the left hand support of the beam as shown in fig.9.5. Total load acting on the beam length PQ is equal to r. dz

Let, shear force at P = F , and shear force at Q = F + dF

Bending moment at P = M and Bending moment at Q = M + dM

For equilibrium condition, ΣSF = 0

$$F + r.dz - (F + dF) = 0$$

$$dF = r.dz$$

$$\boxed{\frac{dF}{dz} = w} \quad \text{-----------------(1)}$$

The above relation shows that ***the rate of change of shear force is the rate of loading per unit length of the beam.***

The force system in fig.9.5 may be simplified as shown in fig.9.5(a). ***The total udl is considered to act as a point load at the middle of the span over which it acts.***

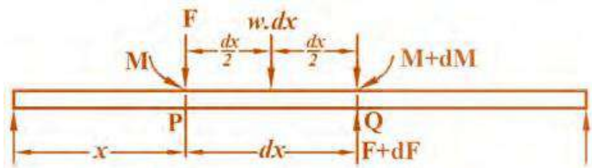


Fig.9.5(a) Relationship between load, SF and BM.

Taking moment of forces and couples about P,

$$-(M + dM) + M - r.\,dx\,\frac{dx}{2} + (F + dF)\,dx = 0$$

$$-M - dM + M - r\,\frac{(dx)^2}{2} + F.\,dx + dF.\,dx = 0$$

Neglecting the small quantities

$$-\,dM + F.\,dz = 0$$

$$dM = F.\,dz$$

$$\boxed{\frac{dM}{dz} = F}$$

The above relation shows that ***the rate of change of bending moment about a section is equal to the SF at that section.***

For maximum bending moment, $\frac{dM}{dz} = 0$ *i.e.* $F = 0$.

Therefore, the bending moment is maximum at a section where shear force is zero.

9.8 Standard cases of loading

1) Cantilever beam with a point load at its free end

Consider a cantilever AB of length l and carrying a point load W at its free end B as shown in the fig.9.6. Consider a section X–X at a distance x from the free end.

Shear force :

SF at B = +W *(Plus sign due to right downward)*

SF at X–X = +W *(∵ There is no load between B and X–X)*

SF at A = +W *(∵ There is no load between X–X and A)*

Bending moment :

Bending moment at X–X = – W z (Minus sign due to hogging)

The bending moment at any section is proportional to the distance of that section from the free end.

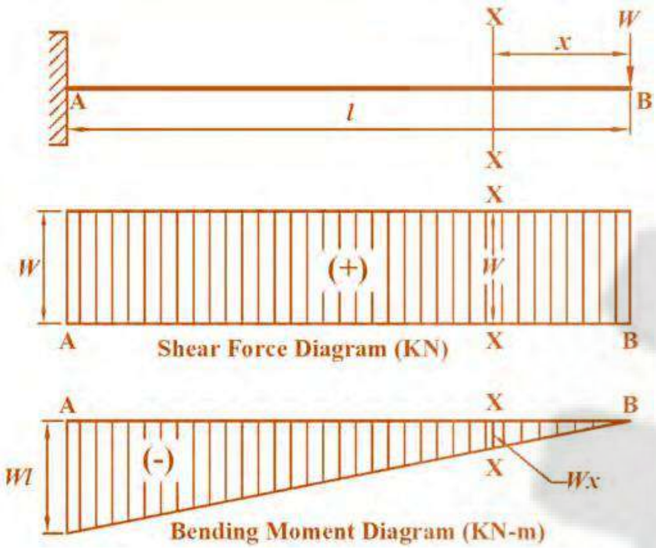


Fig.9.6 Cantilever with a point load at its free end

At B, z = 0; ∴ BM = – W × 0 = 0

At A, z = l; ∴ BM = – W × l = – Wl

2) Cantilever beam with uniformly distributed load

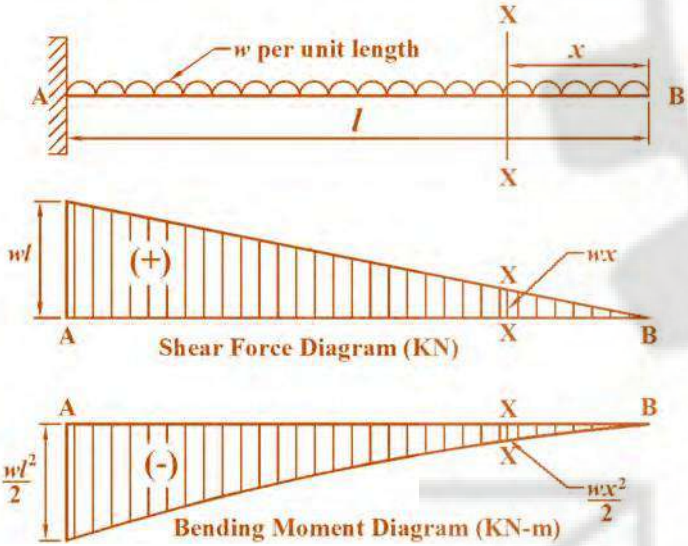


Fig.9.7 Cantilever with uniformly distributed load

Consider a cantilever AB of length l and carrying a uniformly distributed load r per unit length over the entire length of the beam as shown in the fig.9.7. Consider a section X–X at a distance z from the free end.

SF at X–X = +wx *(∵ Plus sign due to right downward)*

Bending moment at X–X $= -rx \times \frac{x}{2} = \frac{-rx^2}{2}$ *(Hogging moment)*

From the above two equations, the shear force varies according to a ***straight line law***, while the bending moment varies according to ***parabolic law***.

Shear force :

At B, x = 0; SF = 0

At X–X , x = x; SF = rx

At A, x = l; SF = rl

Bending moment :

At B, x = 0; BM = 0

At X-X, x = x; $BM = -\frac{rx^2}{2}$

At A, x = l; $BM = -\frac{rl^2}{2}$

3) Simply supported beam with point load at the mid span

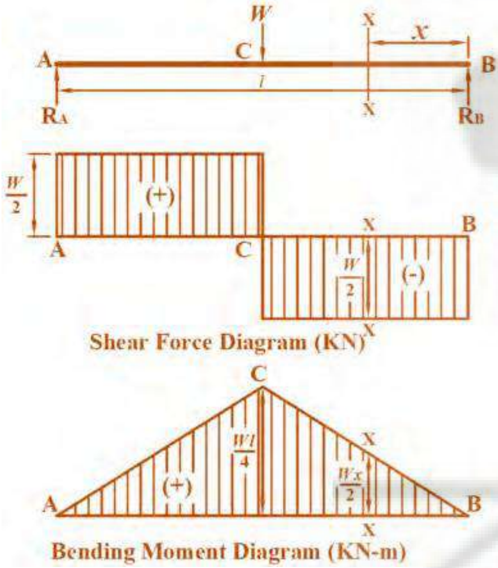


Fig.9.8 Simply supported beam with point load at mid span.

Consider a simply supported beam AB of length l and carrying a point load W at its mid point C as shown in the fig.9.8.

Let R_A and R_B be the reactions at the supports A and B. Taking moment about the support A,

$$R_B \times l = W \times \frac{l}{2}$$

$$R_B = \frac{Wl}{2l} = \frac{W}{2}$$

But, $R_A + R_B = W$

$$R_A = W - \frac{W}{2} = \frac{W}{2}$$

Consider a section X–X at a distance *x* from B.

Shear force :

Shear force at B $= -\frac{W}{2}$ *(∵ Minus sign due to right upward)*

Shear force at X–X $= -\frac{W}{2}$

Shear force remains constant between B and C and is equal to $\frac{W}{2}$

Shear force at C $= -\frac{W}{2} + W = \frac{W}{2}$

Shear force remains constant between C and A and is equal to $\frac{W}{2}$

Shear force at A $= +\frac{W}{2}$

Bending moment :

Bending moment at X–X $= +\frac{W}{2}x$ *(∵ Plus due to sagging)*

At B, z = 0 ; BM = 0

At C, $x = \frac{l}{2}$; $BM = +\frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$

At A, BM = 0

4) Simply supported beam with uniformly distributed load over entire span

Consider a simply supported beam AB of length l and carrying a udl of r per unit length, over the entire length as shown in the fig.9.9.

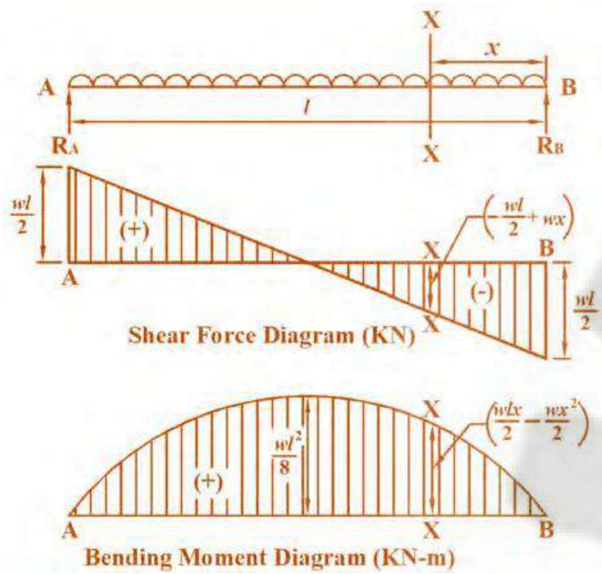


Fig.9.9 Simply supported beam with udl over the entire length

Let R_A and R_B be the reactions at the supports A and B. Taking moment about the support A,

$$R_B \times l = rl \times \frac{l}{2}$$

$$R_B = \frac{rl^2}{2l} = \frac{rl}{2}$$

But, $$R_A + R_B = rl$$

$$R_A = rl - \frac{rl}{2} = \frac{rl}{2}$$

Consider a section X–X at a distance x from B.

Shear force :

Shear force at B $= -\frac{rl}{2}$ *(∵ Minus sign due to right upward)*

Shear force at X–X $= -\frac{rl}{2} + rx$

Shear force at C $\left(x = \frac{l}{2}\right) = -\frac{rl}{2} + \frac{rl}{2} = 0$

Shear force at A$(z = l)$ $= -\frac{wl}{2} + wl = \frac{wl}{2}$

Bending moment :

$$\text{Bending moment at X–X} = R_B z - wz\frac{z}{2} = \frac{wlz}{2} - \frac{wz^2}{2}$$

At B, z = 0; BM = 0

At C, $\left(z = \frac{l}{2}\right)$; $BM = \frac{wl}{2} \times \frac{l}{2} - \frac{w}{2}\left(\frac{l}{2}\right)^2 = \frac{wl^2}{4} - \frac{wl^2}{8} = \frac{wl^2}{8}$

At B (z = l) $BM = \frac{wl^2}{2} - \frac{wl^2}{2} = 0$

9.9 Hints for calculating SF and BM at a section

1) Calculation of shear force

(a) Consider a section at which shear force is to be calculated

(b) Consider all the loads which act either to the right or to the left of the section.

(c) Find the algebraic sum of the loads by using sign conventions for shear force. This sum gives the value of shear force at that section.

2) Calculation of bending moment

(a) Consider a section at which bending moment is to be calculated

(b) Consider all the loads which act either to the right or to the left of the section.

(c) Take moment of these loads about that section.

(d) Find the algebraic sum of the moments by using sign convention of bending moment. This sum gives the value of bending moment at that section.

(e) A concentrated load which passes through the considered section have zero moment about that section.

(f) The bending moment at the free end of a cantilever beam and the two supports of SSB will be zero.

(g) The udl is considered to act as a point load at the middle of the span over which it acts.

9.10 Hints for drawing SF and BM diagrams

1) Shear force diagram

(a) If there is a point load at a section, the shear force line will suddenly increase or decrease by a vertical line.

(b) If there is no load between any two sections, the shear force will remain constant and shear force line will be a horizontal straight line parallel to the base line.

(c) If there is a uniformly distributed load between two sections, the shear force line will be an inclined straight line.

(d) When a point load acts along with a uniformly distributed load, the SF diagram will have two inclined lines separated by a vertical straight line at a point where point load acts.

(e) In a cantilever beam, the maximum shear force will occur at the fixed end. In a simply supported beam, the maximum shear force will occur at the supports.

2) Bending moment diagram

(a) The bending moment line in a region between two point loads will be an inclined straight line.

(b) The bending moment line in a region of udl will be a parabolic line.

9.11 Point of contraflexure

Overhanging beam can be considered as combination of simply supported beam and a cantilever beam. We know that the bending moment in the simply supported beam is positive, whereas the bending moment in the cantilever beam is negative. It is thus known that in an overhanging beam, there will be a point, where the bending moment will change sign from positive to negative and *vice versa*. Such a point, where the bending moment changes sign, is known as a ***point of contraflexure.***

SOLVED PROBLEMS

CANTILEVER BEAMS

Example : 9.1 *(Apr.01)*

A cantilever 2m long carries a point load of 3KN at its free end and another point load of 2KN at a distance of 0.5m from the free end. Draw the shear force and bending moment diagram.

Solution :

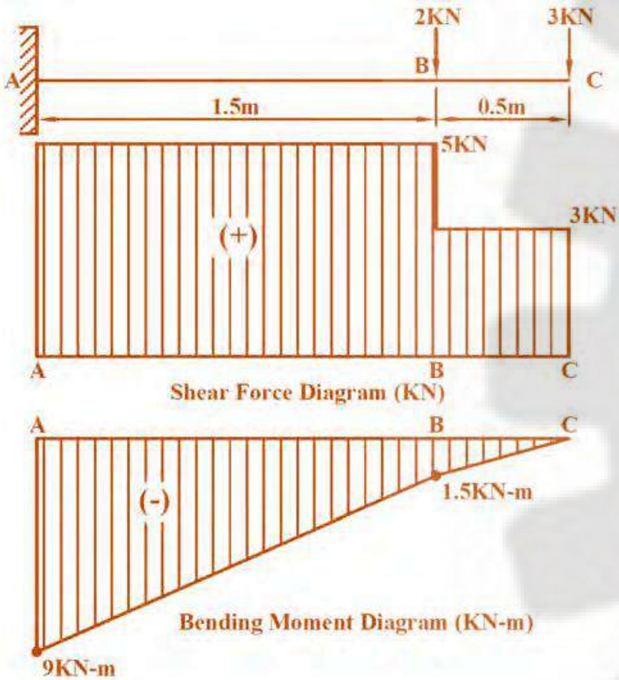


Fig.P9.1 SF and BM diagram [Example 9.1]

Calculation for shear force :

Shear force at C = +3 KN

Shear force at B = +3 + 2 = 5 KN

Shear force at A = +5 KN *(There is no load between B & A)*

Calculation for bending moment :

Bending moment at C = 0

Bending moment at B = −3 × 0.5 = −1.5 KN−m

Bending moment at A = −3 × 2 − 2 × 1.5 = −9 KN−m

Example : 9.2

A cantilever of span 10 m carries point loads of 6KN and 8KN at 4m and 7m from the fixed end. Draw SF and BM diagram.

Solution :

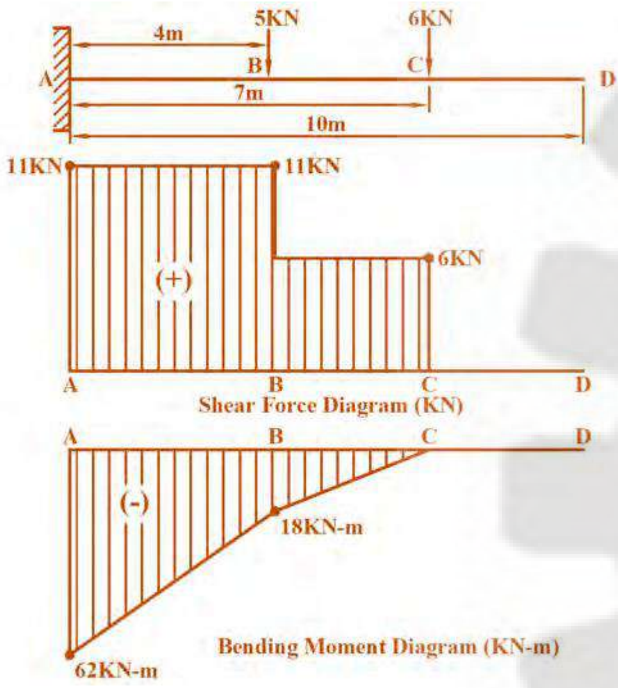


Fig.P9.2 SF and BM diagram [Example 9.2]

Calculation for shear force :

SF at D = 0 *(There is no load)*

SF at C = + 6 KN

SF at B = + 6 + 5 = +11 KN

SF at A = + 11 KN *(∵ There is no load between B and A)*

Calculation for bending moment :

BM at D = 0

BM at C = 0

BM at B = – 6 × 3 = –18 KN–m

BM at A = – 6 × 7 – 5 × 4 = – 62 KN–m

Example : 9.3 *(Apr.89, Oct.96, Oct.03, Oct.12, Apr.17)*

A cantilever 4m long carries a udl of 30KN/m over half of its length adjoining the free end. Draw SF and BM diagrams.

Solution :

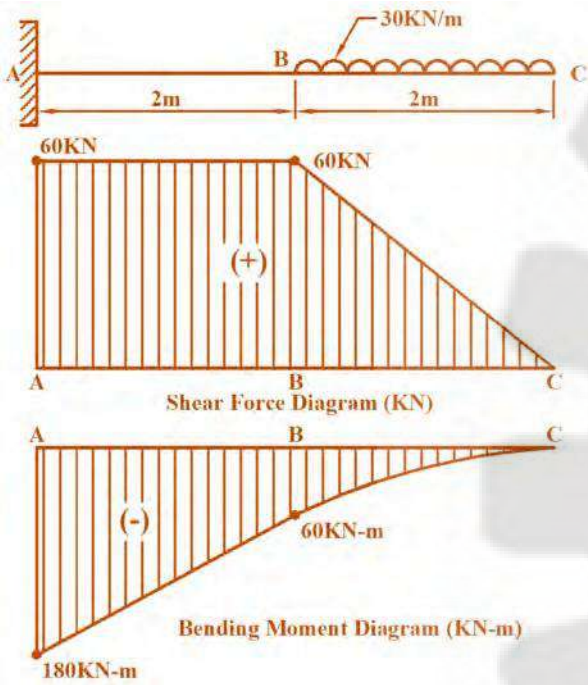


Fig.P9.3 SF and BM diagram [Example 9.3]

Calculation for shear force :

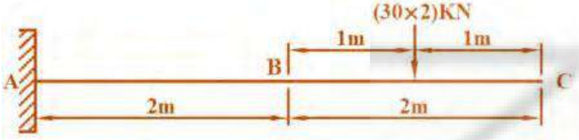
SF at C = 0 (*There is no load)*

SF at B = + 30 × 2 = + 60 KN

SF at A = + 60 KN (*There is no load between B and A)*

Calculation for bending moment :

Note : udl is assumed as a point load acting at the middle of udl span.



BM at C = 0

BM at B $= -30 \times 2 \times \left(\frac{2}{2}\right) = -60$ KN-m

BM at A $= -30 \times 2 \times \left(2 + \frac{2}{2}\right) = -180$ KN-m

Example : 9.4

(Oct.88, Apr.92, Oct.03)

A cantilever of 2m long carries a point load of 20KN at 0.8mm from the fixed end and another point load of 5KN at the free end. In addition a udl of 15KN/m is spread over the entire length of the cantilever. Draw the SF and BM diagrams.

Solution :

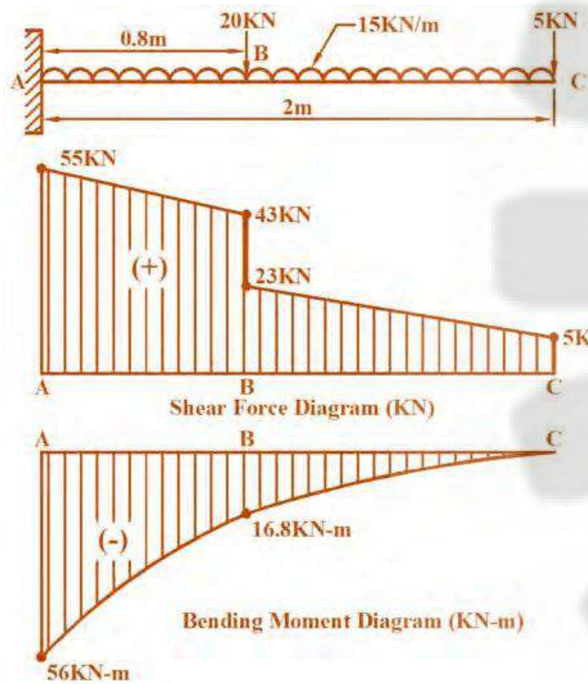


Fig.P9.4 SF and BM diagram [Example 9.4]

Calculation for shear force :

SF at C = + 5 KN

SF at B (Due to udl) $= +5 + (15 \times 1.2) = +23$ KN

SF at B (Due to point load) $= +23 + 20 = +43$ KN

SF at A $= +43 + (15 \times 0.8) = +55$ KN

Calculation for bending moment :

BM at C = 0

BM at B $= -(5 \times 1.2) - \left(15 \times 1.2 \times \frac{1.2}{2}\right) = -16.8$ KN−m

BM at A $= -(5 \times 2) - \left(15 \times 2 \times \frac{2}{2}\right) - (20 \times 0.8) = -56$ KN−m

Example : 9.5

(Oct.92, Apr.13)

Draw the shear force and bending moment diagrams for the loaded beam shown in the fig.P9.5

Solution :

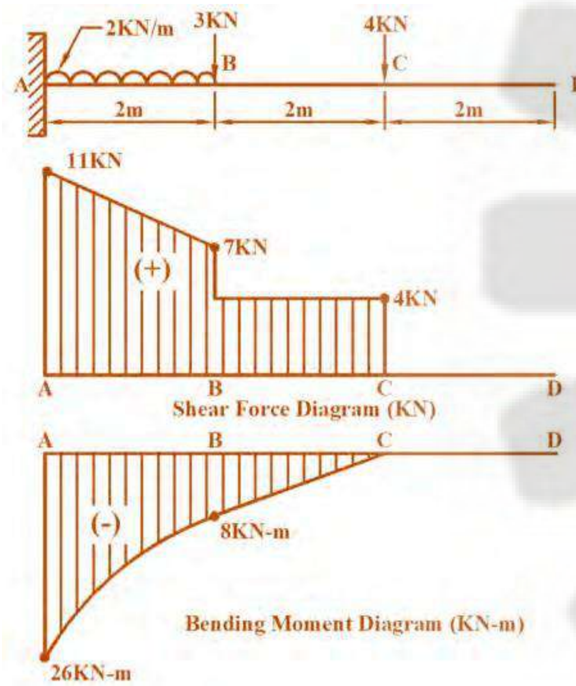


Fig.P9.5 SF and BM diagram [Example 9.5]

Calculation for shear force :

SF at D = 0

SF at C = + 4 KN

SF at B = + 4 + 3 = +7 KN

SF at A = +7 +(2 × 2) = +11 KN

Calculation for bending moment :

BM at D = 0

BM at C = 0

BM at B = – 4 × 2 = –8 KN–m

BM at A = $-(4 \times 4) - (3 \times 2) - \left(2 \times 2 \times \frac{2}{2}\right) = -26$ KN–m

Example : 9.6 *(Apr.93)*

Draw the shear force and bending moment diagrams for the loaded beam shown in the fig.P9.6

Solution :

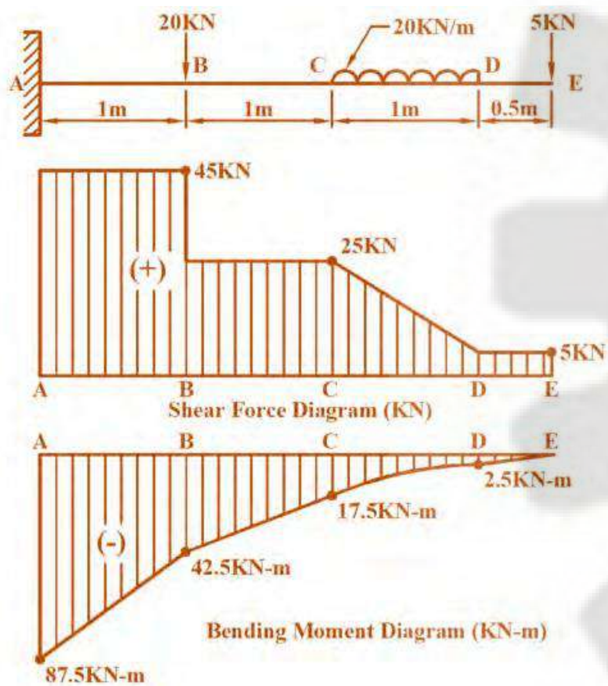


Fig.P9.6 SF and BM diagram [Example 9.6]

Calculation for shear force :

SF at E = + 5 KN

SF at D = + 5 KN

SF at C = + 5 + (20 × 1) = +25 KN

SF at B = +5 +(20 × 1) + 20 = +45 KN

SF at A = + 45 KN (∵ *There is no load between B & A)*

Calculation for bending moment :

BM at E = 0

BM at D = −5 × 0.5 = −2.5 KN−m

BM at C = $-(5 \times 1.5) - \left(20 \times 1 \times \frac{1}{2}\right) = -17.5$ KN−m

BM at B = $-(5 \times 2.5) - \left[20 \times 1 \times \left(1 + \frac{1}{2}\right)\right] = -42.5$ KN−m

BM at A = $-(5 \times 3.5) - \left[20 \times 1 \times \left(2 + \frac{1}{2}\right)\right] - 20 \times 1) = -87.5$ KN−m

SIMPLY SUPPORTED BEAMS

Example : 9.7 *(Apr.97)*

A simply supported beam 5m span carries a point load of 20KN at 2m from left support. Draw the shear force and bending moment diagrams.

Solution :

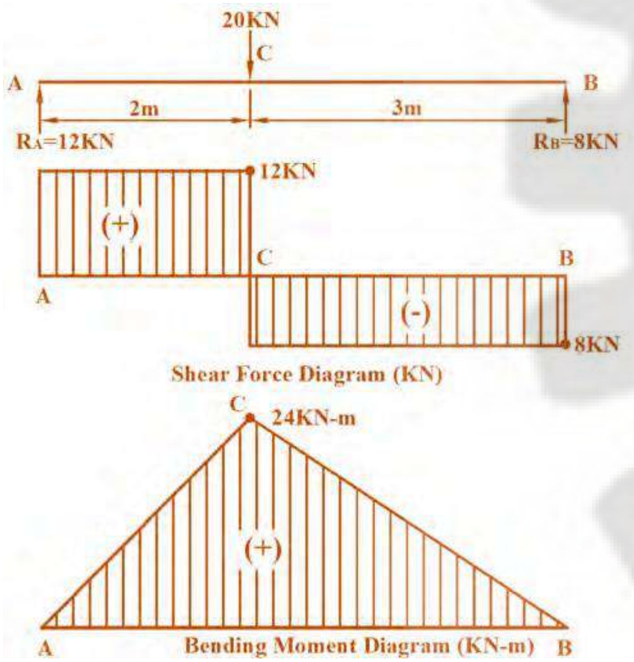


Fig.P9.7 SF and BM diagram [Example 9.7]

Taking moment about

A, $R_B \times 5 = 20 \times 2$

$R_B = \frac{40}{5} =$ 8 KN

But, $R_A + R_B = 20$ KN

$R_A = 20 - R_B = 20 - 8 = 12$ KN

Calculation for shear force :

Shear force at B = – 8 KN (‡ Minus sign due to right upward) Shear force at C = – 8 + 20 = +12 KN

Shear force at A = +12 KN (‡There is no load between C and A)

Calculation for bending moment :

Bending moment at B = 0

Bending moment at C = + 8 × 3 = +24 KN–m

Bending moment at A = +(8 × 5) – (20 × 2) = 0

Example : 9.8 *(Oct.04)*

A simply supported beam of 10m span is loaded with point loads of 20KN, 40KN at 2m and 8m from left support respectively. Draw the shear force and bending moment diagrams.

Solution :

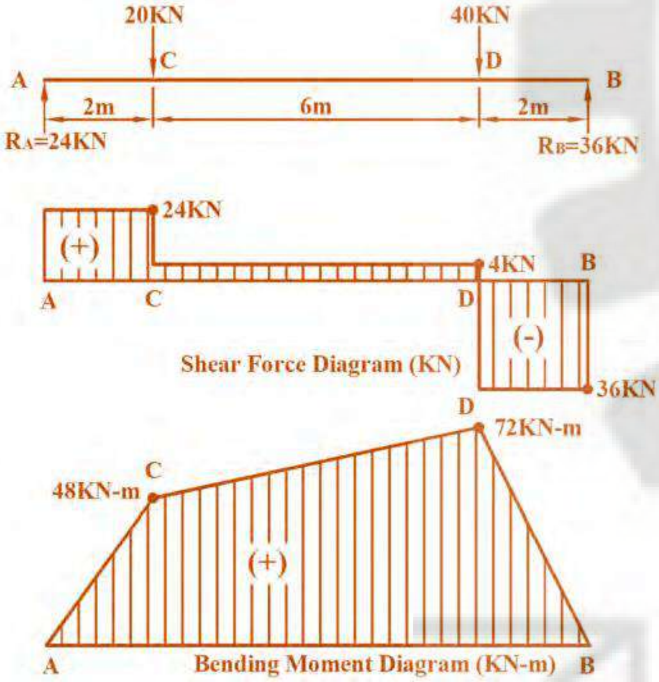


Fig.P9.8 SF and BM diagram [Example 9.8]

Taking moment about A,

$$R_B \times 10 = (40 \times 8) + (20 \times 2) = 360$$

$$R_B = \frac{360}{10} = 36 \text{ KN}$$

But, $R_A + R_B = 60$ KN

$$R_A = 60 - R_B = 60 - 36 = 24 \text{ KN}$$

Calculation for shear force :

SF at B = – 36 KN

SF at D = – 36 + 40 = +4 KN

SF at C = + 4 + 20 = 24 KN

SF at A = + 24 KN (‡There is no load between C and A)

Calculation for bending moment :

BM at B = 0

BM at D = +36 × 2 = +72 KN–m

BM at C = +(36 × 8) – (40 × 6) = +48 KN–m

BM at A = 0

Example : 9.9 *(Apr.88, Oct.03, Oct.16)*

A simply supported beam of effective span 6m carries three point loads of 30KN, 25KN and 40KN at 1m, 3m and 4.5m respectively from the left support. Draw the SF and BM diagrams. Also indicate the maximum value of bending moment.

Solution :

Taking moment about A,

$$R_B \times 6 = (30 \times 1) + (25 \times 3) + (40 \times 4.5) = 285$$

$$R_B = \frac{285}{6} = 47.5 \text{ KN}$$

But, $R_A + R_B = 30 + 25 + 40 = 95$ KN

$$R_A = 95 - R_B = 95 - 47.5 = 47.5 \text{ KN}$$

Calculation for shear force :

SF at B = – 47.5 KN

SF at E = – 47.5 + 40 = – 7.5 KN

SF at D = – 7.5 + 25 = + 17.5 KN

SF at C = + 17.5 + 30 = + 47.5 KN

SF at A = + 47.5 KN (*There is no load between C and A*)

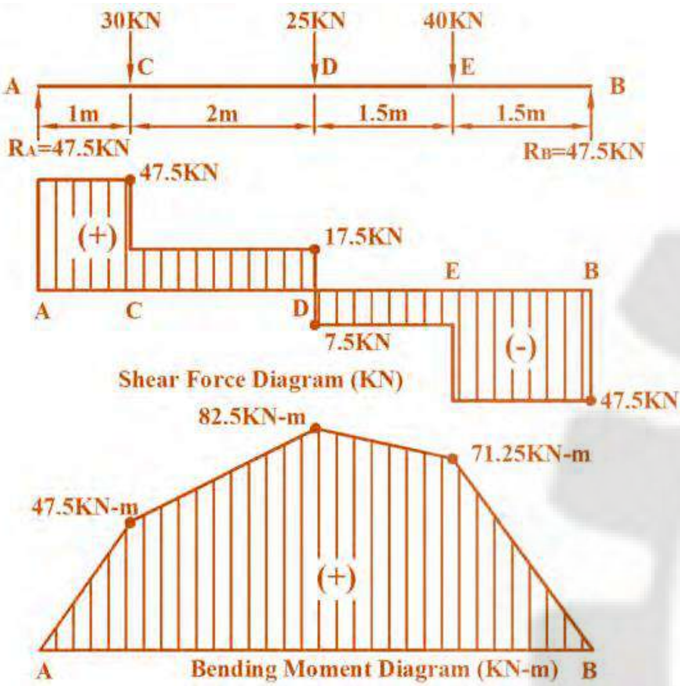


Fig.P9.9 SF and BM diagram [Example 9.9]

Calculation for bending moment :

BM at B = 0

BM at E = +47.5 × 1.5 = +71.25 KN–m

BM at D = +(47.5 × 3) – (40 × 1.5) = + 82.5 KN–m

BM at C = +(47.5 × 5) – (40 × 3.5) – (25 × 2) = + 47.5 KN–m

BM at A = 0

Example : 9.10 *(Oct.96, Oct.17)*

A beam is freely supported over a span of 8m. It carries a point load of 8KN at 2m from the left hand support and a udl of 2KN/m run from the centre up to the right hand support. Construct the SF and BM diagram.

Solution :

Taking moment about A,

$R_B \times 8 = [(2 \times 4) \times \{4 + \frac{4}{2}\}] + (8 \times 2) = 64$

$R_B = \frac{64}{8} = 8$ KN

But, $R_A + R_B = (2 \times 4) + 8 = 16$ KN

$R_A = 16 - R_B = 16 - 8 = 8$ KN

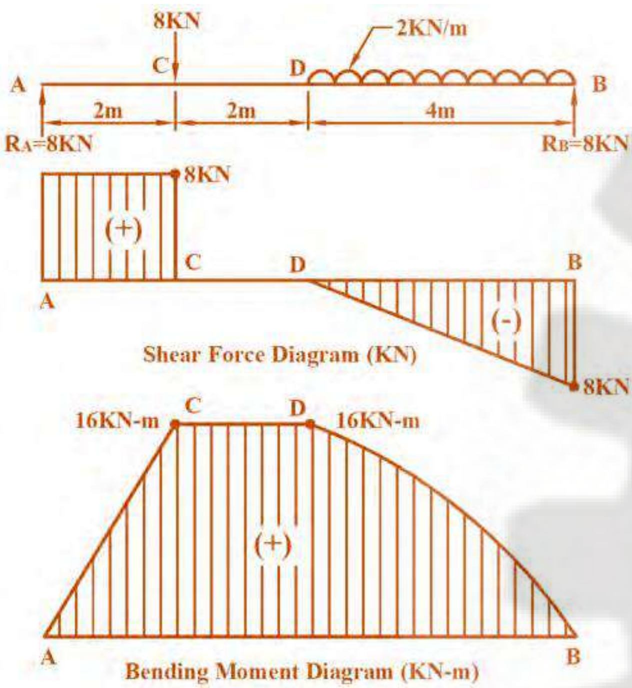


Fig.P9.10 SF and BM diagram [Example 9.10]

Calculation for shear force :

SF at B = − 8 KN

SF at D = − 8 + (2 × 4)= 0 KN

SF at C = 0 + 8 = + 8 KN

SF at A = 8 KN (‡There is no load between C and A)

Calculation for bending moment :

BM at B = 0

BM at D = $+(8 \times 4) - \left(2 \times 4 \times \frac{4}{2}\right) = +16$ KN−m

BM at C = $+(8 \times 6) - \left[2 \times 4 \times \left(2 + \frac{4}{2}\right)\right] = +16$ KN−m

BM at A = 0

Example : 9.11 *(Oct.88, Apr.93, Oct.01, Apr.14, Oct.14, Apr.17)*

A simply supported beam of length 6m carries a udl of 20KN/m throughout its length and a point load of 30KN at 2m from the right support. Draw the shear force and bending moment diagram. Also find the position and magnitude of maximum bending moment.

Solution :

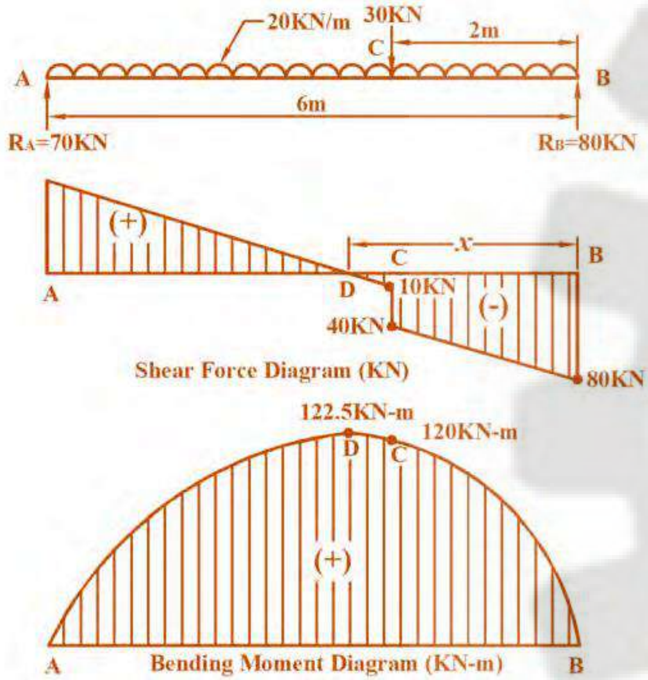


Fig.P9.11 SF and BM diagram [Example 9.11]

Taking moment about A,

$$R_B \times 6 = \left(20 \times 6 \times \frac{6}{2}\right) + (30 \times 4) = 480$$

$$R_B = \frac{480}{6} = 80\ KN$$

But, $R_A + R_B = (20 \times 6) + 30 = 150$ KN

$R_A = 150 - R_B = 150 - 80 = 70$ KN

Calculation for shear force :

SF at B = – 80 KN

SF at C (Due to udl) = – 80 + (20 × 2) = –40 KN

SF at C (Due to point load) = – 40 + 30 = –10 KN

SF at A = –10+(20 × 4) = + 70 KN

Calculation for bending moment :

BM at B = 0

BM at C = $+(80 \times 2) - \left(20 \times 2 \times \frac{2}{2}\right) = +120$ KN–m

BM at A = 0

To find the maximum bending moment :

The bending moment will be maximum at a point where the shear force is equal to zero. Let D be the point at a distance 'z' from B at which shear force is zero.

Shear force at D = – 80 + 20z + 30 = 0

$20z = 50$

$z = \frac{50}{20} = \mathbf{2.5}$

The bending moment will be maximum at a distance **2.5 m** from the right support (B).

Maximum bending moment at D = $+(80 \times 2.5) - (30 \times 0.5) - \left(20 \times 2.5 \times \frac{2.5}{2}\right)$

$= 122.5$ **KN–m**

Example : 9.12 ***(Oct.04, Apr.18)***

A simply supported beam of span 10m carries a udl of 20kN/m over the left half of the span and a point load of 30KN at the mid span. Draw the SFD and BMD. Find also the position and magnitude of maximum bending moment.

Solution :

Taking moment about A,

$R_B \times 10 = (30 \times 5) + \left(20 \times 5 \times \frac{5}{2}\right) = 400$

$R_B = \frac{400}{10} = 40$ KN

But, $R_A + R_B = 30 + (20 \times 5) = 130$ KN

$R_A = 130 - R_B = 130 - 40 = 90$ KN

Calculation for shear force :

SF at B = – 40 KN

SF at C = – 40 + 30 = – 10 KN

SF at A = – 10 +(20 × 5) = + 90 KN

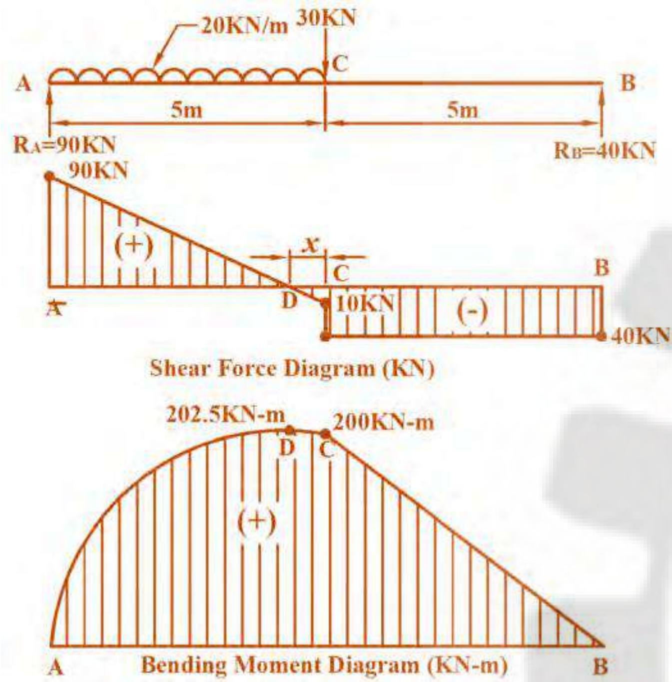


Fig.P9.12 SF and BM diagram [Example 9.12]

Calculation for bending moment :

BM at B = 0

BM at C = +(40 × 5) = +200 KN−m

BM at A = 0

To find the maximum bending moment :

The bending moment will be maximum at a point where the shear force is equal to zero. Let D be the point at a distance 'z' from C at which the shear force is zero.

Shear force at D = − 40 + 30 + 20z = 0

$$20\,z = 10$$

$$z = \frac{10}{20} = \mathbf{0.5}$$

The bending moment will be maximum at a distance **5.5m** from the point B. Maximum bending moment at D

$$= +(40 \times 5.5) - (30 \times 0.5) - \left(20 \times 0.5 \times \frac{0.5}{2}\right) = 202.5 \text{ KN–m}$$

Example : 9.13 *(Apr.01)*

A simply supported beam AB of 8m length carries an udl of 5KN/m for a distance of 4m from the left end support A. The rest of the beam of 4m carries an udl of 10KN/m. Draw SF and BM diagrams.

Solution :

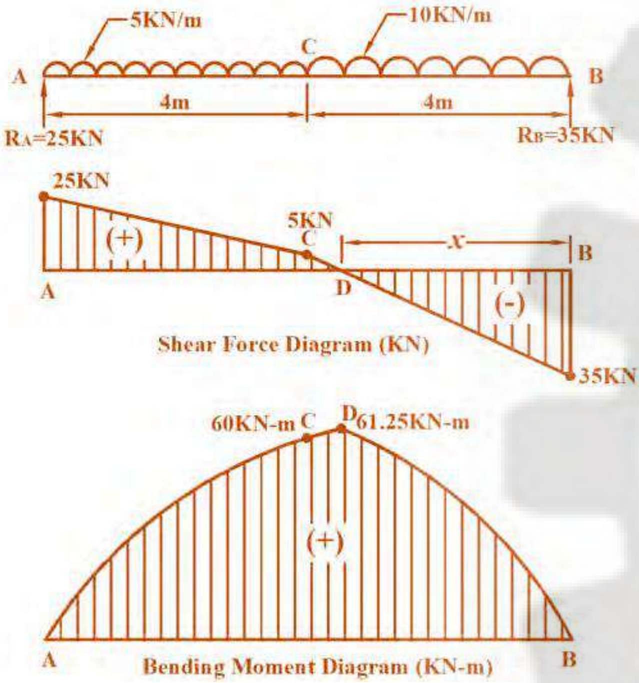


Fig.P9.13 SF and BM diagram [Example 9.13]

Taking moment about A,

$$R_B \times 8 = \left[10 \times 4 \times \left(4 + \frac{4}{2}\right) + 5 \times 4 \times \left(\frac{4}{2}\right)\right] = 280$$

$$R_B = \frac{280}{8} = 35\ KN$$

But, $R_A + R_B = (10 \times 4) + (5 \times 4) = 60$ KN

$R_A = 60 - R_B = 60 - 35 = 25$ KN

Calculation for shear force :

SF at B = – 35 KN

SF at C = – 35 + (10 × 4) = + 5 KN

SF at A = +5 + (5 × 4) = + 25 KN

Calculation for bending moment :

BM at B = 0

BM at C = $+(35 \times 4) - \left(10 \times 4 \times \frac{4}{2}\right) = +60$ KN–m

BM at A = 0

To find the maximum bending moment :

The bending moment will be maximum at a point where the shear force is equal to zero. Let D be the point at a distance 'z' from B at which the shear force is zero.

Shear force at D = – 35 + 10z = 0

$$z = \frac{35}{10} = \mathbf{3.5}$$

The bending moment will be maximum at a distance **3.5m** from the point B. Maximum bending moment at D

$$= +(35 \times 3.5) - \left(10 \times 3.5 \times \frac{3.5}{2}\right) = 61.25 \text{ KN–m}$$

Example : 9.14 *(Oct.94)*

Draw the SF and BM diagrams for the beam shown in the fig.P.9.14 and also calculate the maximum bending moment.

Solution :

Taking moment about A,

$R_B \times 5 = (4 \times 4) + (8 \times 3 \times 2.5) + (2 \times 1) = 78$

$R_B = \frac{78}{5} = 15.6$ KN

But, $R_A + R_B = 4 + (8 \times 3) + 2 = 30$ KN

$R_A = 30 - R_B = 30 - 15.6 = 14.4$ KN

Calculation for shear force :

SF at B = – 15.6 KN

SF at D = – 15.6 + 4 = – 11.6 KN

SF at C*(due to udl)* = – 11.6 + (8 × 3) = +12.4 KN

SF at C*(due to point load)* = +12.4 + 2 = 14.4 KN

SF at A = + 14.4 KN

(There is no load between C and A)

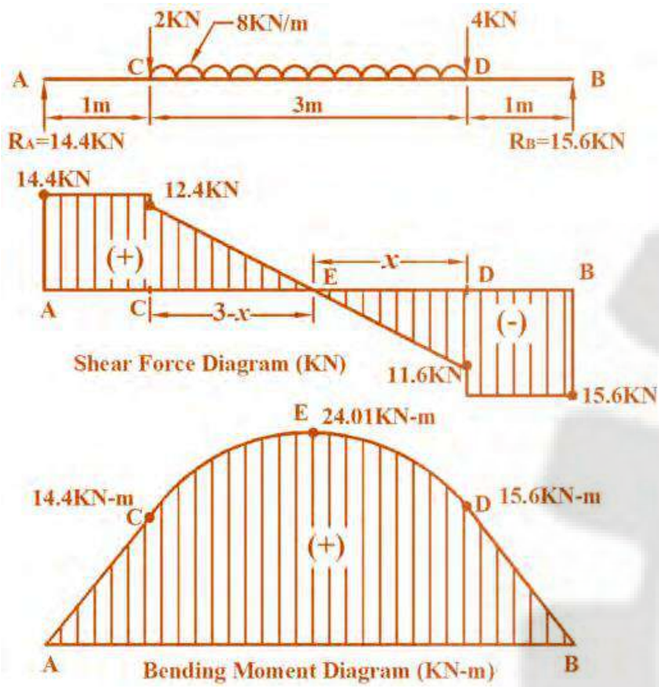


Fig.P9.14 SF and BM diagram [Example 9.14]

Calculation for bending moment :

BM at B $= 0$

BM at D $= +(15.6 \times 1) = +15.6$ KN−m

BM at C $= +(15.6 \times 4) - \left(8 \times 3 \times \frac{3}{2}\right) = 14.4$ KN−m

BM at A $= 0$

To find the maximum bending moment :

The bending moment will be maximum at a point where the shear force is equal to zero. Let E be the point at a distance 'z' from D at which shear force is zero.

Shear force at E $= -15.6 + 4 + 8z = 0$

$z = \frac{11.6}{8} = \mathbf{1.45}$

The bending moment will be maximum at a distance **1.45m** from the point D. Maximum bending moment at E

$$= +(15.6 \times 2.45) - (4 \times 1.45) - \left(8 \times 1.45 \times \frac{1.45}{2}\right) = 24.01 \text{ KN-m}$$

Example : 9.15 *(Oct.91)*

Draw the SF and BM diagrams for the beam shown in the fig.P.9.15 and also calculate the maximum bending moment.

Solution :

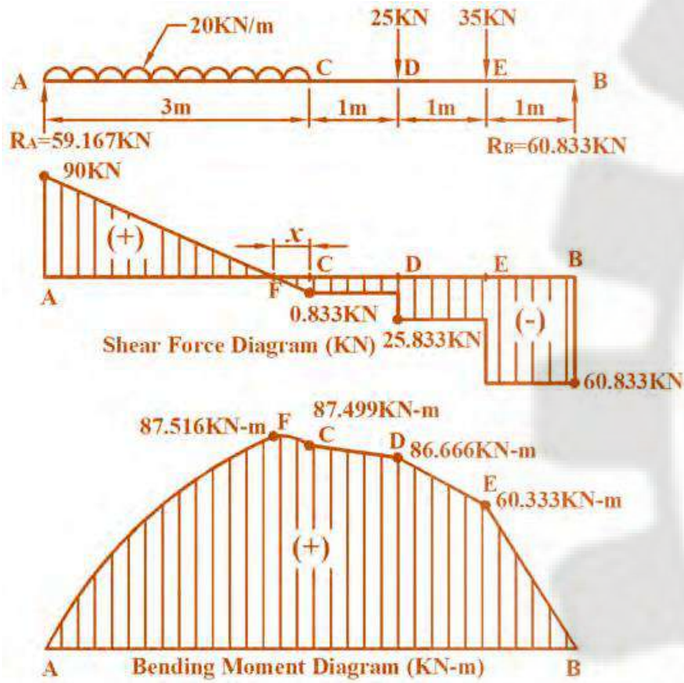


Fig.P9.15 SF and BM diagram [Example 9.15]

Taking moment about A,

$$R_B \times 6 = (35 \times 5) + (25 \times 4) + \left(20 \times 3 \times \frac{3}{2}\right) = 365$$

$$R_B = \frac{365}{6} = 60.833 \text{ KN}$$

But, $R_A + R_B = 35 + 25 + (20 \times 3) = 120$ KN

$R_A = 120 - R_B = 120 - 60.833 = 59.167$ KN

Calculation for shear force :

SF at B = – 60.833 KN

SF at E = – 60.833 + 35 = – 25.83 KN

SF at D = – 25.833 + 25 = –0.833 KN

SF at C = – 0.833 KN *(There is no load between D and C)*

SF at A = 0.833 + (20 × 3) = + 59.167 KN

Calculation for bending moment :

BM at B = 0

BM at E = +(60.833 × 1) = +60.833 KN–m

BM at D = +(60.833 × 2) – (35 × 1) = + 86.666 KN–m

BM at C = +(60.833 × 3) – (35 × 2) – (25 × 1) = + 87.499 KN–m

BM at A = 0

To find the maximum bending moment :

The bending moment will be maximum at a point where the shear force is equal to zero. Let F be the point at a distance 'z' from C at which the shear force is zero.

Shear force at F = – 60.833+ 35 + 25 +20 z = 0

$$z = \frac{0.833}{20} = \mathbf{0.04165}$$

The bending moment will be maximum at a distance **0.04165m** from the point C.

Maximum bending moment at F

= +(60. 833 × 3. 04165)– (35 × 2. 04165) – (25 × 1. 04165) – (20 × 0. 04165 × 0. 04165/2) = + 87. 516 KN–m

Example : 9.16

A simply supported beam of span 7m is subjected to a udl of 10KN/m for 3m from left support and a udl of 5KN/m for 2m from the right support. Draw the SF and BM diagrams. Also calculate the maximum bending moment.

Solution :

Taking moment about A,

$$R_B \times 7 = \left[5 \times 2 \times \left(5 + \frac{2}{2}\right) + 10 \times 3 \times \left(\frac{3}{2}\right)\right] = 105$$

$$R_B = \frac{105}{8} = 15 \text{ KN}$$

But, $R_A + R_B$ = (5 × 2) + (10 × 3) = 40 KN

$R_A = 40 - R_B = 40 - 15 = 25$ KN

Calculation for shear force :

SF at B = – 15 KN

SF at D = – 15 + (5 × 2) = – 5 KN

SF at C = – 5 KN

SF at A = + 25 KN

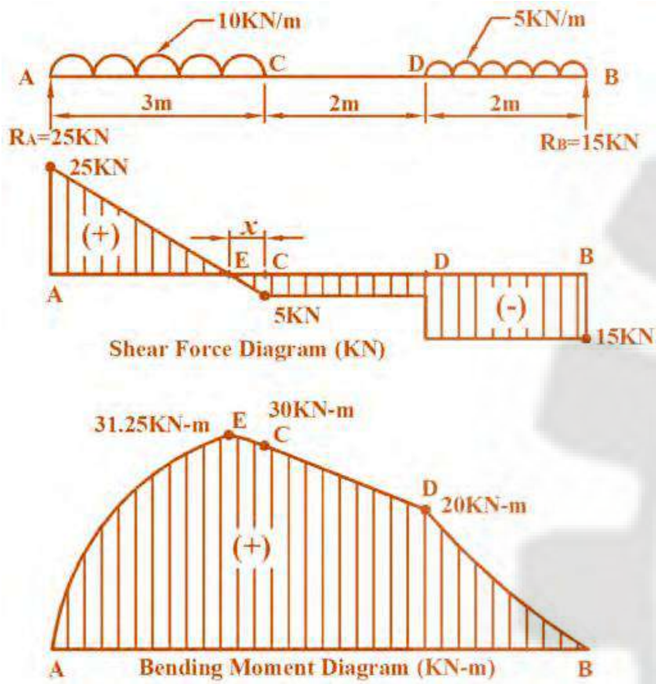


Fig.P9.16 SF and BM diagram [Example 9.16]

Calculation for bending moment :

BM at B = 0

BM at D = $+(15 \times 2) - \left(5 \times 2 \times \frac{2}{2}\right) = +20$ KN–m

BM at C = $+(15 \times 4) - (5 \times 2 \times 3) = +30$ KN–m

BM at A = 0

To find the maximum bending moment :

The bending moment will be maximum at a point where the shear force is equal to zero. Let E be the point at a distance 'z' from C at which the shear force is zero.

Shear force at E = $-15 + (5 \times 2) + 10z = 0$

$z = \frac{5}{10} = \mathbf{0.5}$

The bending moment will be maximum at a distance **0.5m** from the point C. Maximum bending moment at E

$= +(15 \times 4.5) - \left[5 \times 2 \times \left(2.5 + \frac{2}{2}\right)\right] - \left(10 \times 0.5 \times \frac{0.5}{2}\right) = +31.25$ **KN–m**

Unit – V

Chapter 10. THEORY OF BENDING

1. Introduction

When a beam is loaded with some external forces, bending moment and shear forces are set up. The bending moment at a section tends to bend

or deflect the beam and internal stresses are developed to resist this bending. These stresses are called *bending stresses* and the relevant theory is called *theory of simple bending.*

1. Simple bending or pure bending

If a beam tends to bend or deflect only due to the application of constant bending moment and not due to shear force, then it is said to be in a

state c

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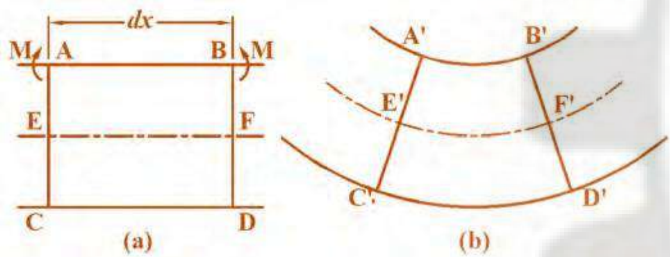

Fig.10.1 Theory of simple bending

Consider a small length dx of simply supported beam subjected to a bending moment M as shown in the fig.10.1(a). Due to the action of the bending moment, the beam as a whole will bend as shown in fig.10.1(b). Due to bending, the length of the beam is changed. Let us consider a top most layer AB and bottom most layer CD. The layer AB is subjected to compression and shortened to A’B’ while the layer CD is subjected to tension and stretched to C’D’.

Let us consider the beam length dx consists of large number of such layers. The length of all the layers are changed due to bending. Some of them may be shortened while some others may be stretched. However, there exists a layer EF in between the top and bottom layers which will retain its original length even after bending. This layer EF which is neither shortened nor stretched is known as the *neutral layer* or *neutral plane.*

10.4 Assumptions made in the theory of simple bending

The following are the assumptions made in the theory of simple bending.

1) The material of the beam is uniform throughout.
2) The material of the beam has equal elastic properties in all directions.
3) The beam material is stressed within elastic limit and thus obeys Hooke's law.
4) The beam material has same value of Young's modulus both in tension and compression.
5) The radius of curvature of the beam is very large when compared with the cross sectional dimensions of the beam.
6) The resultant pull or push on a transverse section of the beam is zero.
7) Each layer of the beam is free to expand or contract independently of the layer, above or below it.
8) The cross section of the beam which is plane and normal before bending will remain plane and normal even after bending.

5. Neutral axis

The line of intersection of the neutral layer with any normal cross–
section of the beam is known as *neutral axis* of that section. It is denoted as N.A. A beam is subjected to compressive stresses on one side of the neutı axis. Therı

5.

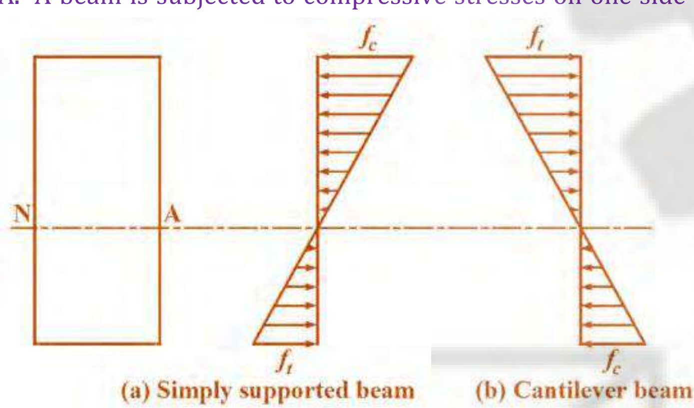

Fig.10.2 Bending stress distribution

There is no stress at the neutral axis. The magnitude of stress at a point is directly proportional to its distance from the neutral axis. The maximum stress taken place at the outer most layer.

In a simply supported beam, compressive stresses are developed above the neutral axis and tensile stresses are developed below the neutral axis. But in cantilever beam, tensile stresses are developed above the neutral axis and compressive stresses are developed below the neutral axis.

7. Moment of resistance

The maximum bending moment that a beam can withstand without failure is called moment of resistance.

From the theory of simple bending, we know that one side of the neutral axis is subjected to compressive stresses and other side of the neutral axis is subjected to tensile stresses. These compressive and tensile stresses form a couple, whose moment must equal to the external moment (M). The moment of this couple which resist the external bending moment is known as moment of resistance.

7) Derivation of flexural formula

a) To prove $\frac{f}{y} = \frac{E}{R}$

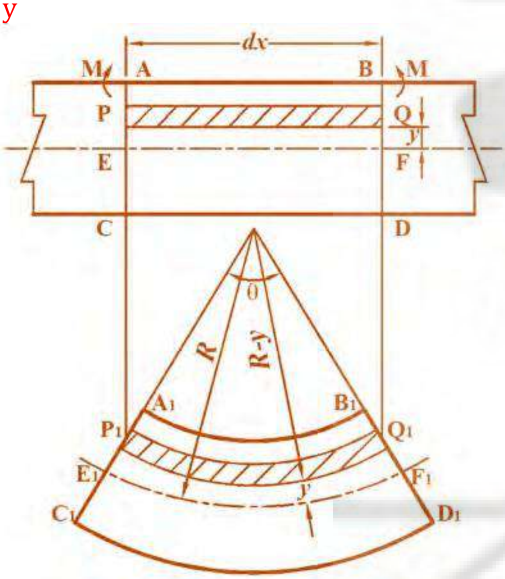

Fig.10.3 Bending stress

Consider a small length dz of a beam subjected to a bending moment as shown in the fig.10.3. As a result of this bending moment, this small length of beam bend into an arc of circle with O as centre.

Let, M = Moment acting at the beam

& = Angle subtended at the centre by the arc and

R = Radius of curvature of the beam

Now consider a length PQ at a distance 'y' from the neutral axis EF. Let this layer be compressed to P_1Q_1 after bending.

We know that, decrease in length of this layer,

$$6l = PQ - P_1Q_1 = R\& - (R - y)\&$$

Strain in the layer, $e = \frac{\text{change in length}}{\text{Original length}} = \frac{y\&}{R\&} = \frac{y}{R}$

If 'f' be the bending stress in the layer, then

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{f}{e}$$

$$f = E \times e = E \times \frac{y}{R}$$

$$\boxed{\frac{f}{y} = \frac{E}{R}}$$

Since E and R for a beam are constant, the bending stress is directly proportional to the distance of the layer from the neutral axis.

$$\therefore \frac{f_1}{y_1} = \frac{f_2}{y_2} = \dots = \frac{f_{maz}}{y_{maz}}$$

b) To prove $\frac{M}{I} = \frac{E}{R} =$ F

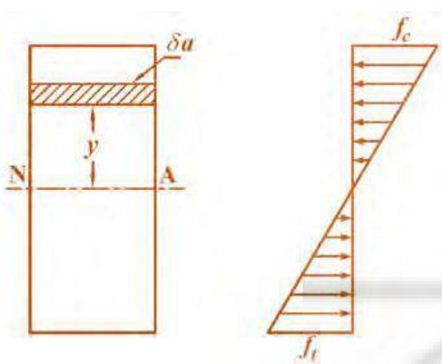

Fig.10.4 Neutral axis

Consider a small elemental area 6a of a beam at a distance 'y' from neutral axis as shown in fig.10.4

Let 'f' be the bending stress in the elemental area.

The force on the elemental area = f × 6a

Moment of this force about neutral axis,

$$6M = f \times 6a \times y \qquad \text{------------ (1)}$$

Substitute, $f = y \times \frac{E}{R}$ in equation (1)

$$6M = \frac{yE}{R} \times 6a \times y = \frac{E}{R} 6a\, y^2$$

By definition, moment of resistance

$$M = Z6M = Z \frac{E}{R} 6a\, y^2 = \frac{E}{R} Z6a\, y^2$$

We know that $Z6a\, y^2$ = *Moment of inertia* of the area of the section about neutral axis i.e. I

$$\therefore M = \frac{E}{R} \times I \text{ (or)}$$

$$\frac{M}{I} = \frac{E}{R} \qquad \text{------------ (2)}$$

Also, $\frac{f}{y} = \frac{E}{R}$ ----------- (3)

Combining the equations (2) and (3)

$$\boxed{\frac{M}{I} = \frac{f}{y} = \frac{E}{R}}$$

The above equation is called *flexural equation.*

10.9 Section modulus

The ratio of moment of inertia about the neutral axis to the distance of the extreme layer from the neutral axis is known as *section modulus* or *modulus of section*.

$$\text{Section modulus} = \frac{\text{Moment of inertial about N.A}}{\text{Distance of extreme layer from N.A}}$$

We know that the maximum bending stress occurs at the outermost layer. Let y_{maz} be the distance of the outermost layer and f_{maz} be the maximum stress.

From the flexural formula, $f_{maz} = \frac{M}{I} \times y_{maz}$ (or)

$$M = f_{maz} \frac{I}{y_{maz}} = f_{maz} \times Z$$

Where Z= Section modulus or modulus of section.

Section modulus of various sections

1) Rectangular section

Consider a rectangular section of width 'b' and depth 'd'.

Moment of inertia about the neutral axis, $I = \frac{bd^3}{12}$

Distance of extreme layer from N.A, $y_{maz} = \frac{d}{2}$

$$\therefore \text{ Section Modulus, } Z = \frac{I}{y_{maz}} = \frac{\frac{bd^3}{12}}{\frac{d}{2}} = \frac{bd^2}{6}$$

2. Circular section

Consider a circular section of diameter 'd'.

Moment of inertia about the neutral axis, $I = \frac{vd^4}{64}$

Distance of extreme layer from N.A, $y_{maz} = \frac{d}{2}$

$$\therefore \text{ Section Modulus, } Z = \frac{I}{y_{maz}} = \frac{\frac{vd^4}{64}}{\frac{d}{2}} = \frac{vd^3}{32}$$

1.10 Strength and stiffness of beam

Strength : The moment of resistance offered by the beam is known as *strength* of a beam.

We know that, moment of resistance, $M = f \times Z$

From the above relation, it is known that, for a given value of bending stress, the moment of resistance depends upon the section modulus. Therefore, if the value of Z is greater, the beam will be strong. This ideal is put into practice, by providing beam of I –section, where the flanges alone withstand almost all the bending stress.

Stiffness : The resistance offered by a beam against deflection from its original straight condition is known as *stiffness* of the beam.

SOLVED PROBLEMS

Example : 10.1

A steel wire of 5mm diameter is bent into a circular shape of 5m radius. Determine the maximum stress induced in the wire. Take $E = 2 \times 10^5 N/mm^2$.

Given : Diameter of the steel wire, d = 5 mm

Radius of circular shape, R = 5 m = 5000 mm

Young's modulus, $E = 2 \times 10^5 N/mm^2$

To find : 1) The maximum stress induced, f_{max}

Solution :

Distance of extreme layer from neutral axis (N.A.)

$$y_{max} = \frac{d}{2} = \frac{5}{2} = 2.5 \text{ mm}$$

We know that, $\frac{f_{max}}{y_{max}} = \frac{E}{R}$

$$f_{max} = \frac{E}{R} \times y_{max} = \frac{2 \times 10^5 \times 2.5}{5000} = 100 \text{ N/mm}^2$$

Result : 1) The maximum stress induced in the wire, f_{maz} = **100** N/mm^2

Example : 10.2

(Apr.93, Oct.02)

A steel rod 100mm diameter is to be bent into circular shape. Find the maximum radius of curvature which it should be bent so that stress in the steel should not exceed 120 N/mm^2***.Take*** $E = 2 \times 10^5 N/mm^2$.

Given : Diameter of the steel rod, d = 100 mm

Maximum bending stress, $f_{max} = 120 \text{ N/mm}^2$

Young's modulus, $E = 2 \times 10^5 N/mm^2$

To find : 1) The radius of curvature, R

Solution :

Distance of extreme layer from neutral axis (N.A.)

$$y_{max} = \frac{d}{2} = \frac{100}{2} = 50 \text{ mm}$$

We know that, $\frac{f_{max}}{y_{max}} = \frac{E}{R}$

$$R = \frac{E}{f_{max}} \times y_{max} = \frac{2 \times 10^5 \times 50}{120} = 83333 \text{ mm}$$

Result : 1) The radius of curvature, R = **83333 mm**

Example : 10.3

A metallic rod of 10mm diameter is bent into a circular form of radius 6m. If the maximum bending stress developed in the rod is 125 N/mm^2, find the value of Young's modulus for the rod material.

Given : Diameter of the rod, d = 10 mm

Maximum bending stress, $f_{max} = 125\ N/mm^2$

Radius of curvature, R = 6 m = 6000 mm

To find : 1) Young's modulus, E

Solution :

Distance of extreme layer from neutral axis (N.A.) $y_{max} = \frac{d}{2} = \frac{100}{2} = 50$ mm

We know that, $\frac{f_{max}}{y_{max}} = \frac{E}{R}$

$E = \frac{R}{y_{max}} \times f_{max} = \frac{6000 \times 125}{5} = 1.5 \times 10^5\ N/mm^2$

Result : 1) Young's modulus of the material, $E = 1.5 \times 10^5\ N/mm^2$

Example : 10.4

(Oct.01)

Determine the resisting moment of a timber beam rectangular in section 125mm × 250mm, if the permissible bending stress is 8 N/mm^2.

Given : Maximum bending stress, $f_{max} = 8\ N/mm^2$

Width of the beam, b = 125 mm

Depth of the beam, d = 250 mm

To find : 1) Resisting moment, M

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{125 \times 250^3}{12} = 1.6276 \times 10^8\ mm^4$

Distance of extreme layer from neutral axis (N.A.) $y_{max} = \frac{d}{2} = \frac{250}{2} = 125$ mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$M = \frac{f_{max}}{y_{max}} \times I = \frac{8 \times 1.6276 \times 10^8}{125} = 10.417 \times 10^6$ **N-mm**

Result : 1) Resisting moment, $M = 10.417 \times 10^6$ **N-mm**

Example : 10.5

(Oct.92, Oct.14, Oct.15)

***A simply supported beam is 300mm wide and 400mm deep. Determine the bending stress at 40mm above N.A, if the maximum bending stress is 15**N/mm²*.

Given : Width of the beam, b = 300 mm

Depth of the beam, d = 400 mm

Distance of layer from the N.A, y_1 = 40 mm

Maximum bending stress, $f_{max} = 15\ N/mm^2$

To find : 1) Bending stress at a distance 40mm above the N.A, f_1

Solution :

Distance of extreme layer from neutral axis (N.A.)

$$y_{max} = \frac{d}{2} = \frac{400}{2} = 200\ mm$$

We know that, $\frac{f_1}{y_1} = \frac{f_{max}}{y_{max}}$

$$f_1 = \frac{f_{max}}{y_{max}} \times y_1 = \frac{15}{200} \times 40 = 3\ N/mm^2$$

Result : 1) Bending stress at a distance 40mm above N.A, $f_1 = 3\ N/mm^2$

Example : 10.6

(Oct.88, Oct.91, Oct.12, Oct.13)

A rectangular beam 200mm deep and 100mm wide is simply supported over a span of 8m and carries a central point load of 25KN. Determine the maximum stress in the beam. Also calculated the value of longitudinal fibre stress at a distance of 25mm from the surface of the beam.

Given : Width of the beam, b = 100 mm

Depth of the beam, d = 200 mm

Length of the beam, l = 8m = 8000 mm

Central point load, W = 12 KN =12 × 10^3 N

To find : 1) Maximum bending stress, f_{max}

2) Bending stress at 25mm from the surface of the beam, f_1

Solution :

$$\text{Moment of inertia, } I = \frac{bd^3}{12} = \frac{100 \times 200^3}{12} = 66.667 \times 10^6\ mm^4$$

Distance of extreme layer from neutral axis (N.A.) $y_{max} = \frac{d}{2} = \frac{200}{2} = 100$ mm

In case of simply supported beam subjected to a central point load, Maximum bending moment, $M = \frac{W\,l}{4}$

$$= \frac{25 \times 10^3 \times 8000}{4} = 50 \times 10^6 \text{ N-mm}$$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$f_{max} = \frac{M}{I} \times y_{max} = \frac{50 \times 10^6 \times 100}{66.667 \times 10^6} = \boxed{75 \text{ N/mm}^2}$$

To find the bending stress at 25mm from the surface of the beam :

The distance of layer from N.A, $= y_1 = 100 - 25 = 75$ mm

$$\frac{f_1}{y_1} = \frac{f_{max}}{y_{max}}$$

$$f_1 = \frac{f_{max}}{y_{max}} \times y_1 = \frac{75}{100} \times 75 = \boxed{56.25 \text{ N/mm}^2}$$

Result : 1) The maximum bending stress, $f_{maz} = 75$ N/mm^2

2) Bending stress at 25mm from surface of beam, $f_1 =$ **56.25** N/mm^2

Example : 10.7

(Apr.14, Apr.15, Oct.15)

A simply supported beam of rectangular cross section carries a central load of 25 KN over a span of 6m. The bending stress should not exceed 7.5 N/mm^2***. The depth of the section is 400mm. Calculate the necessary width of the section.***

Given : Central point load, $W = 25$ KN $= 25 \times 10^3$ N

Length of the beam, $l = 6$m $= 6000$ mm

Bending stres, $f_{max} = 7.5$ N/mm^2

Depth of the beam, $d = 150$ mm

To find : 1) Width of the beam, b

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{b \times 400^3}{12} = 5.333 \times 10^6 \text{ b mm}^4$

Distance of extreme layer from neutral axis (N.A.) $y_{max} = \frac{d}{2} = \frac{400}{2} = 200$ mm

In case of simply supported beam subjected to a central point load,

Maximum bending moment, $M = \frac{W\,l}{4}$

$= \frac{25 \times 10^3 \times 6000}{4} = 37.5 \times 10^6$ N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$\frac{37.5 \times 10^6}{5.333 \times 10^6\, b} = \frac{200}{7.5}$

$b = \frac{37.5 \times 10^6 \times 200}{7.5 \times 5.333 \times 10^6} =$ **187. 5 mm**

Result : 1) Width of the beam, b = 187. 5 **mm**

Example : 10.8

(Apr.87, Oct.89, Oct.04, Apr.17)

A rectangular beam 300mm deep is simply supported over a span of 4m. What udl per metre, the beam may carry if the bending stress is not to exceed 120 N/mm^2***. Given*** $I = 8 \times 10^6\ mm^4$.

Given : Depth of the beam, d = 300 mm

Length of the beam, l = 4m = 4000 mm

Maximum bending stress, $f_{max} = 120\ N/mm^2$

Moment of inertia, $I = 8 \times 10^6\ mm^4$

To find : 1) The of udl per metre, r

Solution :

Distance of extreme layer from neutral axis (N.A.)

$y_{max} = \frac{d}{2} = \frac{300}{2}$

= 150 mm

In case of simply supported beam subjected to a udl,

Maximum bending moment, $M = \frac{rl^2}{8} = \frac{r \times 4000^2}{8} = 2 \times 10^6 r$ N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$\frac{2 \times 10^6 r}{8 \times 10^6} = \frac{120}{150}$

$r = \frac{120 \times 8 \times 10^6}{150 \times 2 \times 10^6} = 3.2$ N/mm = **3. 2 KN/m**

Result : 1) The udl per metre, w =**3.2 KN/m**

Example : 10.9 *(Apr.13)*

A rectangular beam 60mm wide and 150mm deep is simply supported over a span of 4m. If the beam is subjected to a uniformly distributed load of 4.5KN/m, find the maximum bending stress induced in the beam.

Given : Width of the beam, b = 60 mm

Depth of the beam, d = 150 mm

Length of the beam, l = 4m = 4000 mm

Uniformly distributed load, r = 4.5 KN/m = 4.5 N/mm

To find : 1) Maximum bending stress, f_{max}

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{60 \times 150^3}{12} = 16.875 \times 10^6 \text{ mm}^4$

Distance of extreme layer from neutral axis (N.A.)

$y_{max} = \frac{d}{2} = \frac{150}{2} = 75 \text{ mm}$

In case of simply supported beam subjected to a udl,

Maximum bending moment, $M = \frac{rl^2}{8} = \frac{4.5 \times 4000^2}{8} = 9 \times 10^6 \text{ N-mm}$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$f_{max} = \frac{M}{I} \times y_{max} = \frac{9 \times 10^6 \times 75}{16.875 \times 10^6} = 40 \text{ N/mm}^2$

Result : 1) Maximum bending stress induced, f_{maz} = **40** N/mm²

Example : 10.10

A timber beam of rectangular section supports a load of 20KN uniformly distributed over a span of 3.6m. If depth of the beam section is twice the width and maximum stress is not to exceed 7N/mm², find the dimension of the beam section.

Given : Total load, W = 20 KN = 20×10^3 N

Length of the beam, l = 3.6 m = 3600 mm

Depth of the beam, d = 2 × width of the beam (b)

Maximum bending stress, $f_{max} = 7 \text{ N/mm}^2$

To find : 1) Depth of the beam, d 2) Width of the beam, b

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{b \times (2b)^3}{12} = 0.667 \, b^4$

Distance of extreme layer from neutral axis (N.A.) $y_{max} = \frac{d}{2} = \frac{2b}{2} = b$

In case of simply supported beam subjected to a udl,

Maximum bending moment, $M = \frac{rl^2}{8} = \frac{Wl}{8}$

$$= \frac{20 \times 10^3 \times 3600}{8} = 9 \times 10^6 \text{ N-mm}$$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$\frac{9 \times 10^6}{0.667\, b^4} = \frac{7}{b}$$

$$7 \times 0.667\, b^4 = 9 \times 10^6 \times b$$

$$b^3 = \frac{9 \times 10^6}{7 \times 0.667} = 1.9276 \times 10^6$$

b = 124.453 mm

Result : 1) Depth of the beam, d = **248.906 mm**
2) Width of the beam, b = **124.453 mm**

Example : 10.11

(Oct.02)

A beam of T–section flange 150mm × 50mm, web thickness 50mm, overall depth 200mm and 10m long is simply supported (with flange uppermost) and carries a central point load of 10KN. Determine the maximum fibre stress in the beam.

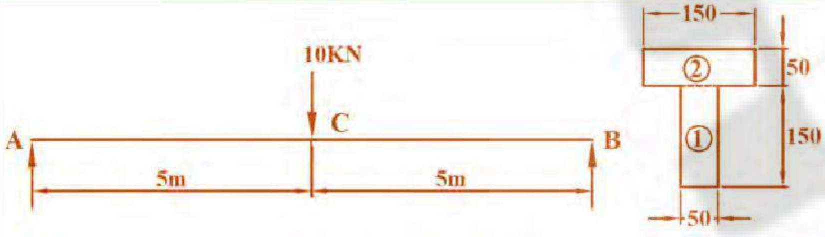

Fig.P10.1 Maximum BM in T–sectional beam [Example. 10.11]

Given : Central point load, W = 10 KN = 10×10^3 N
Length of the beam, l = 10m = 10×10^3 mm

To find : 1) Maximum fibre stress, f_{max}

Solution :

In case of simply supported beam subjected to a point load,

Maximum bending moment, $M = \frac{Wl}{4}$

$$= \frac{10 \times 10^3 \times 10 \times 10^3}{4} = 25 \times 10^6 \text{ N-mm}$$

Distance of extreme layer from N.A, $y_{max} = \bar{Y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2}$

$$= \frac{(50 \times 150 \times 75) + (150 \times 50 \times 175)}{(50 \times 150) + (150 \times 50)} = 125 \text{ mm}$$

Moment of inertia of the section about an axis passing through the centroid and parallel to the bottom face,

$$I = [I_{g1} + a_1h_1^2] + [I_{g2} + a_2h_2^2]$$

$$= [\frac{50 \times 150^3}{12} + (50 \times 150)(125 - 75)^2]$$

$$+ [\frac{150 \times 50^3}{12} + (150 \times 50)(125 - 175)^2]$$

$$= 32.8125 \times 10^6 + 20.3125 \times 10^6 = 53.125 \times 10^6 \text{ mm}^4$$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$f_{max} = \frac{M}{I} \times y_{max} = \frac{25 \times 10^6 \times 125}{53.125 \times 10^6} = 58.824 \text{ N/mm}^2$$

Result : 1) Maximum fibre stress, f_{maz} = **58.824** N/mm^2

Example : 10.12

(Oct.90)

A simply supported beam of span 6m carries uniformly distributed load of intensity 40KN/m over half of the span. The cross section of the beam is symmetrical I–section with following dimensions: Overall depth=300mm, flange width=120mm, flange thickness=25mm, web thickness=12mm. Evaluate the maximum bending stress induced in the beam.

To find : 1) Maximum bending stress induced in the beam, f_{max}

Solution :

Let R_A and R_B be the reactions at the supports of the beam.

Taking moment about A,

$$R_B \times 6 = (40 \times 3 \times 3/2) = 180$$

$$R_B = \frac{180}{6} = 30 \text{ KN}$$

But, $R_A + R_B = (40 \times 3) = 120$ KN

$R_A = 120 - RB = 120 - 30 = 90$ KN

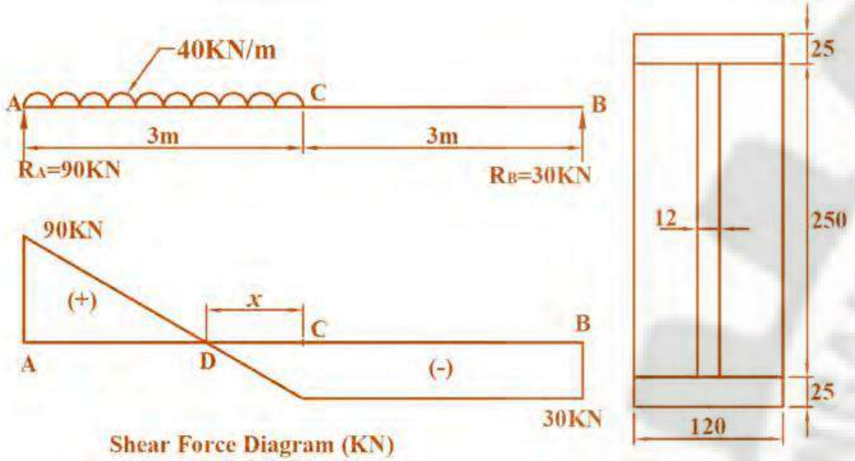

Fig.P10.2 Maximum BM in I–sectional beam [Example. 10.12]

The shear force diagram for the beam is shown in the fig.P10.2. The bending moment will be maximum at a point where the shear force is equal to zero. Let D be the point at a distance 'x' from the point C at which the shear force is zero.

Shear force at D $= -30 + 40x = 0$

$$x = \frac{30}{40} = 0.75 \text{ m}$$

Maximum bending moment at D

$= +(30 \times 3.75) - (40 \times 0.75 \times 0.75/2)$

$= 101.25$ KN–m $= 101.25 \times 10^6$ N– mm

Moment of inertia of the section about an axis passing through the centroid and parallel to the bottom face,

$$I = \left[\frac{120 \times 300^3}{12}\right] - \left[\frac{108 \times 250^3}{12}\right] = 1.294 \times 10^8 \text{ mm}^4$$

Distance of extreme layer from neutral axis (N.A.)

$$y_{max} = \bar{Y} = \frac{300}{2} = 150 \text{ mm}$$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$f_{max} = \frac{M}{I} \times y_{max} = \frac{101.25 \times 10^6 \times 150}{1.294 \times 10^8} =$ **117.369 N/mm²**

Result : 1) Maximum bending stress, f_{maz} = **117.369** N/mm²

Example : 10.13

(Apr.01, Oct.03, Oct.18)

A wooden beam of rectangular section 100mm × 200mm is simply supported over a span of 6m. Determine the udl it may carry if the bending stress is not to exceed 7.5N/mm². Estimate the concentrated load it may carry at the centre of the beam with the same permissible stress.

Given : Width of the beam, b = 100 mm

Depth of the beam, d = 200 mm

Length of the beam, l = 6m = 6000 mm

Maximum bending stress, f_{max} = 7.5 N/mm²

To find : 1) The udl over the entire span, r

2) The point load at the centre for the same stress, W

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{100 \times 200^3}{12} = 66.667 \times 10^6$ mm⁴

Distance of extreme layer from neutral axis (N.A.) $y_{max} = \frac{d}{2} = \frac{200}{2} = 100$ mm

(a) In case of simply supported beam subjected to a udl

Maximum bending moment, $M = \frac{rl^2}{8} = \frac{r \times 6000^2}{8} = 4.5 \times 10^6 r$ N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$\frac{4.5 \times 10^6 r}{66.667 \times 10^6} = \frac{7.5}{100}$$

$$r = \frac{7.5 \times 66.667 \times 10^6}{100 \times 4.5 \times 10^6} = 1.1111 \text{ N/mm} = \boxed{1.1111 \text{ KN/m}}$$

(b) In case of simply supported beam subjected to a point load,

Maximum bending moment, $M = \frac{Wl}{4} = \frac{W \times 6000}{4} = 1500$ W N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$\frac{1500 W}{66.667 \times 10^6} = \frac{7.5}{100}$$

$$W = \frac{7.5 \times 66.667 \times 10^6}{100 \times 1500} = 3333.35\ N = \boxed{3.3333\ KN}$$

Result : 1) The udl over the entire span, w = **1.1111 KN/m**

2) The point load at the centre of the beam, W = **3.3333 KN**

Example : 10.14

(Oct.93, Apr.13)

The moment of inertia of a rolled steel joist girder of symmetrical section about N.A is $2460 \times 10^4 mm^4$***. The total depth of the girder is 240mm. Determine the longest span when simply supported such that the beam would carry a udl of 5KN/m run and the bending stress should not to exceed 120*** N/mm^2***.***

Given : Moment of inertia, $I = 2460 \times 10^4 mm^4$

Depth of the girder, d = 240 mm

Load, r = 6 KN/m = 6 N/mm

Maximum bending stress, $f_{max} = 120\ N/mm^2$

To find : 1) The longest span, l

Solution :

Distance of extreme layer from neutral axis (N.A)

$$y_{max} = \frac{d}{2} = \frac{240}{2} = 120\ mm$$

In case of simply supported beam subjected to a udl,

$$\text{Maximum bending moment, } M = \frac{rl^2}{8} = \frac{6 \times l^2}{8} = 0.75\, l^2$$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$\frac{0.75\, l^2}{2460 \times 10^4} = \frac{120}{120} = 1$$

$$l^2 = \frac{2460 \times 10^4}{0.75} = 32.8 \times 10^6$$

$$l = \sqrt{32.8 \times 10^6} = 5727.128\ mm = \boxed{5.727\ m}$$

Result : 1) The longest span, l = **5.727 m**

Example : 10.15

(Oct.92, Oct.94, Oct.12)

Find the dimensions of a timber joist span 10m to carry a brick wall 0.2m thick and 4m height if the weight of the brick wall is 19 KN/mm^3 ***and the maximum permissible stress is limited to 8*** N/mm^2***. The depth of the joist is to be twice its width.***

Given : Thickness of the wall, $t = 0.2$ m = 200 mm

Height of the wall, $h = 4$m = 4000 mm

Length of the wall, $l = 10$ m = 10000 mm

Weight of the brick wall = 19 KN/mm^3

Depth of the joist, d = 2 × Width of the joist (b)

Maximum bending stress, $f_{max} = 8\ N/mm^2$

To find : 1) Width of joist, b 2) Depth of joist, d

Solution :

Volume of the brick wall over full length,

V=Length×thickness×height

$= 10 \times 0.2 \times 4 = 8\ m^3$

Total weight of the wall over full length, W = 19 × 8 = 152 KN

Load on the brick wall per unit length,

$$r = \frac{152}{10} = 15.2\ KN/m = 15.2\ N/mm$$

Distance of extreme layer from neutral axis (N.A.)

$$y_{max} = \frac{d}{2} = \frac{2b}{2} = b$$

$$\text{Moment of inertia, } I = \frac{bd^3}{12} = \frac{b \times (2b)^3}{12} = 0.667\ b^4$$

In case of simply supported beam subjected to a udl,

$$\text{Maximum bending moment, } M = \frac{rl^2}{8} = \frac{15.2 \times 10000^2}{8} = 1.9 \times 10^8\ N\text{-mm}$$

$$\text{We know that, } \frac{M}{I} = \frac{f_{max}}{y_{max}}$$

$$\frac{1.9 \times 10^8}{0.667\ b^4} = \frac{8}{b}$$

$$8 \times 0.667\ b^4 = 1.9 \times 10^8 \times b$$

$$b^3 = \frac{1.9 \times 10^8}{8 \times 0.667} = 35.607 \times 10^6$$

b = 328.98 mm ≈ **330 mm**

d = 2 × b = 2 × 330 = **660 mm**

Result : 1) Width, b = 330 mm 2) Depth, d = 660 mm

Example : 10.16 (Oct.96, Apr.04, Apr.05, Oct.17)

A cast iron water pipe 450 mm bore and 20 mm thick is supported at two points 6 m apart. Assuming each span as simply supported, find the maximum stress in the metal when (a) the pipe is running full (b) the pipe is empty. Specific weight of cast iron is 70 KN/mm^3 ***and that of water is 9.81***KN/mm^3.

Given : Inside diameter of pipe, d_2 = 450 mm

Thickness of the pipe, t = 20 mm

Length of the pipe, l = 6 m = 6000 mm

Specific weight of cast iron = 70 KN/mm^3 = 70×10^{-6} N/mm^3

Specific weight of water = 9.81KN/mm^3 = 9.81×10^{-6}N/mm^3

To find : 1) Maximum stress in the pipe when it is running full, f_{max}

2) Mmaximum stress in the pipe when it is empty, f_{max}

Solution :

Outside diameter of pipe, $d_1 = d_2 + 2t = 450 + (2 \times 20) = 490$ mm

Cross sectional area of pipe, $A_1 = \frac{\pi}{4}(d_1^2 - d_2^2)$

$$= \frac{\pi}{4}(490^2 - 450^2) = 29531 \text{ mm}^2$$

Weight of the pipe per unit length, $r_1 = A_1 \times$ Sp. rt. of pipe

$$= 29531 \times 70 \times 10^{-6} = 2.067 \text{ N/mm}$$

Cross sectional area of the water section,

$$A_2 = \frac{\pi}{4} \times d_2^2 = \frac{\pi}{4} \times 450^2 = 1.5904 \times 10^5 \text{ mm}^2$$

Weight of water per unit length, $r_2 = A_2 \times$ Sp. rt. of rater

$$= 1.5904 \times 10^5 \times 9.81 \times 10^{-6} = 1.56 \text{ N/mm}$$

(a) When the pipe is running full

Total weight per unit length, $r = r_1 + r_2 = 2.067 + 1.56 = 3.627$ N/mm

In case of simply supported beam subjected to a udl,

Maximum bending moment, $M = \frac{rl^2}{8}$

$$= \frac{3.627 \times 6000^2}{8} = 16.3215 \times 10^6 \text{ N-mm}$$

Distance of extreme layer from neutral axis (N.A.)

$$y_{max} = \frac{d_1}{2} = \frac{490}{2} = 245 \text{ mm}$$

Moment of inertia, $I = \frac{\pi}{64}\left(d_1^4 - d^4\right)$

$= \frac{\pi}{64}\left(490^4 - 450^4\right) = 8.169 \times 10^8 \text{ mm}^4$

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$f_{max} = \frac{M}{I} \times y_{max} = \frac{16.3215 \times 10^6 \times 245}{8.169 \times 10^8} = \boxed{4.895 \text{ N/mm}^2}$

(b) When the pipe is empty, only pipe weight is considered.

Weight per unit length, $r = r_1 = 2.067$ N/mm

In case of simply supported beam subjected to a udl,

Maximum bending moment, $M = \frac{rl^2}{8}$

$= \frac{2.067 \times 6000^2}{8} = 9.3015 \times 10^6$ N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$f_{max} = \frac{M}{I} \times y_{max} = \frac{9.3015 \times 10^6 \times 245}{8.169 \times 10^8} = \boxed{2.79 \text{ N/mm}^2}$

Result : 1) Stress in the pipe when it is running full, f_{maz} = **4.895** N/mm²

2) Stress in the pipe when it is empty, f_{maz} = **2.79** N/mm²

CANTILEVER BEAMS

Example : 10.17 *(Oct.92, Apr.13, Apr.14)*

A cantilever of span 1.5m carries a point load of 5KN at the free end. Find the modulus of section required, if the bending stress is not to exceed 150 N/mm²**.**

Given : Load at the free end, W = 5 KN = 5000 N

Length of the beam, l = 1.5 m = 1500 mm

Maximum bending stress, $f_{max} = 150$ N/mm²

To find : 1) Section modulus, Z

Solution :

In case of cantilever subjected to a point load at the free end,

Maximum bending moment, M = Wl = 5000 × 1500 = 7.5 × 10⁶ N-mm

Section modulus, $Z = \frac{M}{f_{max}} = \frac{7.5 \times 10^6}{150} = \boxed{50000 \text{ mm}^3}$

Result : 1) Section modulus, Z = 50000 mm³

Example : 10.18 *(Apr.90, Oct.16)*

A cantilever beam of span 2m carries a point load of 600N at the free end. If the cross–section of the beam is rectangular 100mm wide and 150mm deep, find the maximum bending stress induced.

Given : Length of the beam, l = 2 m = 2000 mm

Load at the free end, W = 600 N

Width of the beam, b = 100 mm

Depth of the beam, d = 150 mm

To find : 1) Maximum bending stress, f_{max}

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{100 \times 150^3}{12} = 28.125 \times 10^6 \ mm^4$

Distance of extreme layer from neutral axis (N.A.)

$y_{max} = \frac{d}{2} = \frac{150}{2} = 75$ mm

In case of cantilever subjected to a point load at the free end,

Maximum bending moment, M = Wl = 600 × 2000 = 1.2×10^6 N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$f_{max} = \frac{M}{I} \times y_{max} = \frac{1.2 \times 10^6 \times 75}{28.125 \times 10^6} =$ **3. 2 N/mm²**

Result : 1) Maximum bending stress, f_{maz} = **3.2 N/mm²**

Example : 10.19

A cantilever beam is rectangular in section having 80mm width and 120mm depth. If the cantilever is subjected to a point load of 6KN at the free end and the bending stress is not to exceed 40N/mm², find the span of the cantilever beam.

Given : Width of the beam, b = 80 mm

Depth of the beam, d = 120 mm

Point load, W = 6 KN = 6000 N

Maximum bending stress, $f_{max} = 40 \ N/mm^2$

To find : 1) Span of the beam, l

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{80 \times 120^3}{12} = 11.52 \times 10^6\ mm^4$

Distance of extreme layer from neutral axis (N.A.)

$y_{max} = \frac{d}{2} = \frac{120}{2} = 60\ mm$

In case of cantilever subjected to a point load at the free end,

Maximum bending moment, M = Wl = 6000 l

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$\frac{6000\ l}{11.52 \times 10^6} = \frac{40}{60}$

$l = \frac{40 \times 11.52 \times 10^6}{6000 \times 60} = 1280\ mm = \boxed{1.28\ m}$

Result : 1) Span of the beam, l = **1.28 m**

Example : 10.20

A square beam 20mm × 20mm in section and 2m in long is supported at the ends. The beam fails when a point load of 400N is applied at the centre of the beam. What udl per metre will break a cantilever of the same material 40mm width and 60mm deep and 3m long.

(i) Simply supported beam

Given : Width of the beam, b = 20 mm

Depth of the beam, d = 20 mm

Length of the beam, l = 2m = 2000 mm

Central point load, W = 400 N

To find : 1) Maximum bending stress, f_{max}

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{20 \times 20^3}{12} = 1.333 \times 10^4\ mm^4$

Distance of extreme layer from neutral axis (N.A.)

$y_{max} = \frac{d}{2} = \frac{20}{2} = 10\ mm$

In case of simply supported beam subjected to a point load,

Maximum bending moment, $M = \frac{Wl}{4} = \frac{400 \times 2000}{4} = 2 \times 10^5$ N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$f_{max} = \frac{M}{I} \times y_{max} = \frac{2 \times 10^5 \times 10}{1.222 \times 10^4} = \boxed{150\ N/mm^2}$$

Result : 1) Maximum bending stess, $f_{maz} = 150\ N/mm^2$

(ii) Cantilever beam

Given : Width of the beam, b = 40 mm

Depth of the beam, d = 60 mm

Length of the beam, l = 3m = 3000 mm

To find : 1) Safe udl spread over the entire span, r

Solution :

Moment of inertia, $I = \frac{bd^3}{12} = \frac{40 \times 60^3}{12} = 7.2 \times 10^5\ mm^4$

Distance of extreme layer from neutral axis (N.A.)

$y_{max} = \frac{d}{2} = \frac{60}{2} = 30\ mm$

For the same material, the bending stress should be equal

∴ Maximum bending stress in the beam, $f_{max} = 150\ N/mm^2$

In case of cantilever beam subjected to a udl over entire span,

Maximum bending moment, $M = \frac{rl^2}{2} = \frac{w \times 3000^2}{2} = 4.5 \times 10^6\ r$ N-mm

We know that, $\frac{M}{I} = \frac{f_{max}}{y_{max}}$

$$\frac{4.5 \times 10^6\ r}{7.2 \times 10^5} = \frac{150}{30}$$

$$r = \frac{150 \times 7.2 \times 10^5}{30 \times 4.5 \times 10^6} = 0.8\ N/mm = \boxed{0.8\ \mathbf{KN/m}}$$

Result : 1) Safe udl spread over the entire span, w = **0.8 KN/m**

Example : 10.21 *(Oct.95)*

A beam of I–section 300mm × 150mm has flanges 20mm thick and web 13mm thick. Compare its flexural strength with that of a rectangular section of the same weight and same material, when the depth being twice the width.

Solution :

Area of I–section = (300 × 20) + (13 × 110) + (300 × 20)

= 13430 mm^2

Moment of inertia of the I–section,

$$I = \left[\frac{300 \times 150^3}{12}\right] - \left[\frac{(300 - 13) \times 110^3}{12}\right] = 52.542 \times 10^6 \text{ mm}^4$$

The section is symmetrical about X-X and Y–Y axis.

$$\therefore y_{max} = \bar{Y} = \frac{150}{2} = 75 \text{ mm}$$

Section modulus of I section, $Z_1 = \frac{I}{y_{max}}$

$$= \frac{52.542 \times 10^6}{75} = 7.0056 \times 10^5 \text{ mm}^3$$

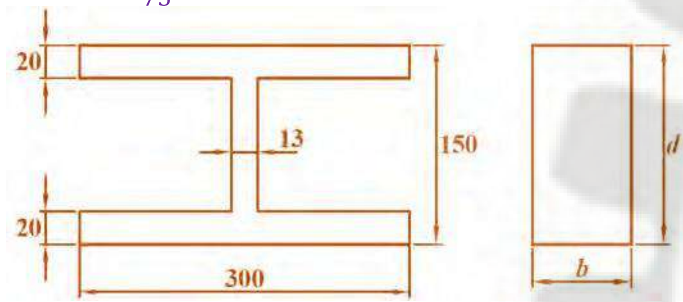

Fig.P10.3 Comparison of flexural strength [Example. 10.21]

Let, b = Width of the required rectangular section

d = Depth of the required rectangular section

Then, d = 2b

For same weight of two beams made of same material, the area of two beams must be equal.

∴ Area of I section = Area of rectangular section

$$13430 = bd = b(2b) = 2b^2$$

$$b^2 = \frac{13430}{2} = 6715$$

$$b = \boxed{81.945 \text{ mm}}$$

$$d = 2b = 2 \times 81.945 = \boxed{163.89 \text{ mm}}$$

Section modulus of rectangular section, $Z_2 = \frac{bd^2}{6}$

$$= \frac{81.945 \times 163.89^2}{6} = 3.668 \times 10^5 \text{ mm}^3$$

The strength of the beam is proportional to its section modulus.

$$\therefore \frac{\text{Flexural strength of I beam}}{\text{Flexural strength of rectangular beam}} = \frac{Z_1 \times E_1}{Z_2 \times E_2} = \frac{Z_1}{Z_2} \quad (\because \text{For same material, } E_1 = E_2)$$

$$= \frac{7.0056 \times 10^5}{3.668 \times 10^5} = \boxed{1.9099}$$

Result : 1) The ratio of flexural strength of two beams = **1.9099**

Example : 10.22

Compare the weights of two beams of same material and of equal flexural strengths, one being circular solid section and other being hollow circular section. The internal diameter being 7/8 of the external diameter.

Solution :

Let, D = Diameter of the solid beam

d_1 = External diameter of the hollow beam

d_2 = Internal diameter of the hollow beam

Then, $d_2 = \frac{7}{8} d_1 = 0.875\, d_1$

Area of solid beam $= \frac{\pi}{4} D^2$

Area of hollow beam $= \frac{\pi}{4}\left(d_1^2 - d_2^2\right)$

$$= \frac{\pi}{4}\left[d_1^2 - (0.875\, d_1)^2\right]$$

$$= \frac{\pi}{4}\left[d_1^2 - 0.765625\, d_1^2\right] = \frac{\pi}{4} \times 0.234375\, d_1^2$$

Section modulus of solid beam, $Z_1 = \frac{\pi}{32} D^3$

Section modulus of hollow beam, $Z_2 = \frac{\pi}{32}\left[\frac{d_1^4 - d_2^4}{d_1}\right]$

$$= \frac{\pi}{32 \times d_1}\left[d_1^4 - (0.875 d_1)^4\right]$$

$$= \frac{\pi}{32 \times d_1}\left[d_1^4 - 0.5862\, d_1^4\right]$$

$$= \frac{\pi}{32 \times d_1} \times 0.4138\, d_1^4 = \frac{\pi}{32} \times 0.4138\, d_1^3$$

Since both the beams have the same flexural strength, the section modulus of both the beams must be equal.

$\therefore Z_1 = Z_2$

$$\frac{\pi}{32} \times D^3 = \frac{\pi}{32} \times 0.4138\, d_1^3$$

$$D^3 = 0.4138\, d_1^3$$

Taking cube root on both sides,

$$D = 0.7452\ d_1$$

Weight of two beams are proportional to their cross sectional areas.

$$\frac{\text{Weight of solid beam}}{\text{Weight of hollow beam}} = \frac{\text{Area of solid beam}}{\text{Area of hollow beam}}$$

$$= \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} \times 0.234375\ d_1^{\,2}}$$

$$= \frac{(0.7452\ d_1)^2}{0.234375\ d_1^{\,2}}$$

$$= \frac{0.5553\ d_1^{\,2}}{0.234375\ d_1^{\,2}}$$

$$= \boxed{2.369}$$

Result : 1) The ratio of weight of solid and hollow beams = **2.369**